Wavelet Theory Sets Out the Welcome Mat

By Barry A. Cipra

The Greek mythological figure Procrustes was hardly the hospitable host. True, he would invite the road-weary traveler into his humble home. But Procrustes had only one bed, and he insisted his visitors fit the accommodations. Too tall? He'd saw off your feet. Too short? He'd hammer out your legs.

The hero Theseus finally passed through and gave Procrustes a dose of his own medicine. But even Theseus couldn't kill an archetype; Procrustes' approach to problem solving pops up in all kinds of places.

Even in applied mathematics.

Take Fourier analysis—please. For the last two centuries, every function on the road to analysis has spent the night at the Sine & Cosine Inn. It's a much nicer place than Procrustes' old shack—very modern, with elaborate theoretical plumbing, easy access from the interstate, and express (FFT) check-out. But it's still got only one bed: $e^{2\pi ix}$.

So beware the turn-down maid. . . .

Mathematicians and their scientific companions have put up with the inflexible house rules of Fourier analysis, in part because the subject really does have tremendous problem-solving power, but also because there weren't any attractive alternatives. But that's beginning to change.

Ingrid Daubechies

A growing number of researchers from many disciplines are touting a new technique in mathematical analysis—wavelets.

Could this just be Procrustes by another name? No. Instead of forcing the function to fit the furniture, wavelets go out of their way to shrink, stretch, and otherwise accommodate the analytic needs of the tired traveler. Developers of the new theory see wavelets as a significant advance over Fourier analysis—the first since Fourier opened to the public in 1822.

"Sines and cosines are a terrible way to represent functions," says Victor Wickerhauser, a wavelet theorist at Yale University. Ingrid Daubechies, who has done pioneering work on wavelets at AT&T Bell Laboratories, has a somewhat kinder assessment: "For some problems, they're the very best tool. But there are many problems . . . for which wavelet analysis might be a better tool."

The reservation desk has certainly been busy. Wavelet theory was the subject of an invited address by Daubechies and two minisymposia at the SIAM Annual Meeting in Chicago this year, and Daubechies gave a CBMS lecture series on wavelets at the University of Lowell in June. *SIAM Review* has already published two survey papers on wavelets. The National Science Foundation, the Air Force Office of Scientific Research, and the Defense Advanced Research Projects Agency are funding research in the area. And perhaps most tellingly, a growing cadre of PhD students are writing their dissertations on wavelets.

Mother Wavelets

What are wavelets, and why have they suddenly appeared on the scene? Like the Fourier transform, wavelets decompose functions into coefficients assigned to fundamental building blocks, from which the original function can be reconstructed. But whereas Fourier analysis builds everything out of sines and cosines, wavelet theory relies on translations and dilations of a suitably chosen "mother wavelet."

The mother wavelet can be virtually any function. (There is a technical constraint that, interestingly, is most easily stated in terms of the Fourier transform.) Most often, however, the mother wavelet is a well-localized "blip"—which may already suggest an application or two. The rest of the building blocks are formed by translating the mother wavelet by unit steps and by contracting or dilating it by factors of 2. This gives a two-parameter transform: one for the distance translated and one for the power of 2.

Stephane Mallat and Sifen Zhong at the Courant Institute used a maxima chain representation (*left*) based on wavelet analysis to compress data from a digitized image. The reconstructed image (*right*) is virtually indistinguishable from the original.

This approach means that wavelet theory can "zoom in" on details of a function in a way that Fourier analysis cannot. Because the sine and cosine functions extend all along the real line, Fourier coefficients contain only "global," not local, information about a function. In particular, transient behavior—the attack of a musical note, for instance—is difficult to see in the Fourier transform. But wavelets are custom-made for analysis of this kind; a blip in a function shows up as a blip in the wavelet transform.

Take a violin concerto, for example. How does Fourier analysis cope with the time-varying function that we perceive as music? In short, it listens to the entire piece and then averages out all of the sounds. The frequencies it finds important may have little to do with the notes that were actually played. And if someone in the audience coughs in the middle of the performance, the accidental sound will reverberate throughout the Fourier transform.

Wavelets, on the other hand, are more akin to musical notation. In essence, each wavelet represents a particular note played at a particular time. (The resemblance is so close, in fact, that one group in France is investigating the use of wavelets to produce printed scores directly from live music.) And minor disruptions remain localized, which makes them easier to edit out.

The use of building blocks other than sine and cosine is hardly new, and many of the ideas in wavelet theory have gone by other names. The Haar basis, for example, in which the mother wavelet takes the value +1 from 0 to $\frac{1}{2}$, and −1 from $\frac{1}{2}$ to 1, is a system of wavelets that dates back as far as 1911. What's new is the recognition that there is a single, powerful theory that brings many ideas together.

Wavelets were introduced as such in France in the early 1980s by Jean Morlet, a geophysicist, and Alexander Grossman, a mathematical physicist, who originally called them "wavelets of constant shape." The mathematical theory took off in 1985, when Yves Meyer, also in France, constructed the first orthogonal system

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of smooth wavelets. (The Haar basis is an orthogonal system, but it is hardly smooth.) In a sense Meyer's result was a failure—he had set out to prove there was no system of smooth orthogonal wavelets. Soon mathematicians were producing such systems of wavelets at will.

In 1986, Meyer and Stephane Mallat developed a theory of "multiresolution analysis" that encompassed the existing systems and provided a natural framework for the theory of wavelet approximations. And in 1987, Daubechies weighed in with a construction of orthogonal, compactly supported wavelets (the size of the support, in Daubechies's theory, grows linearly with the degree of smoothness).

"People often ask, 'What is the best wavelet?' " Daubechies says of the plethora of systems. "There is no answer, because it depends on your problem. There's no magical recipe. You have to think, you have to know what your problem is."

Applications Galore

Although the name may suggest little more than a gentle ripple, wavelets are making a big splash—and not just in the climate-controlled aquarium of theoretical mathematics, but in the shark-infested waters of engineering and physics as well. Wavelet theory seemingly has something for everyone: an elegant theoretical structure for pure mathematicians and wide-ranging applications for those with one or more feet on the ground.

Daubechies calls wavelets an exciting subject: "Never before have I had contacts with people from so many different fields and found that they all were interested. And because everybody has a different way of looking at it, you have all these ideas brewing together and it's very fertile for everybody concerned. It's a very nice laboratory for showing that applications can have interest for pure mathematicians and vice versa."

Much of the interest is in applications to signal processing, and especially to data compression. Stephane Mallat and Sifen Zhong, working at the Courant Institute of Mathematical Sciences, have developed a two-dimensional wavelet transform that compresses the data from a digitized image and still reproduces a pleasing picture. Their algorithm reconstructs one- and two-dimensional functions from the local maxima of the wavelet transform. It seems to be especially well suited to edge detection and characterization, which is still a problem for computer vision.

Ronald Coifman and Victor Wickerhauser at Yale University are investigating speech synthesis using what they call "wavelet packets," a modification that gives extra flexibility in the choice of orthogonal system. Their algorithm searches a library of possibilities for the "best" wavelet representation of a given function. According to Wickerhauser, their algorithm was originally designed to minimize the number of coefficients in the transform above a preset threshold, but they soon realized that they were really minimizing the information entropy of the transform.

Coifman is also working with Gregory Beylkin, a mathematician at Schlumberger Doll Research, on applications of wavelets within numerical analysis itself. They have found that wavelet transforms can "sparsify" the matrices associated with the operators that crop up in many problems in mathematical physics, much as the Fourier transform diagonalizes convolution operators. In effect, if a matrix is viewed literally as a two-dimensional image, the wavelet transform once again compresses the image. With fewer nonzero terms to worry about, calculations can proceed much more rapidly.

There are many other applications in the work as well, from synthetic music to seismic exploration to improvements in medical imaging. So far none of the applications is ready for market, and many ideas that sound exciting now may fall by the wayside. But several patents are in the works, and wavelet researchers—along with the funding agencies—are convinced that wavelet theory, as the name suggests, is the wave of the future.

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