Evolution of a Fundamental Theorem on Quantum Entropy

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Overview

- History of strong subadditivity of quantum entropy (quick)
  Equivalent formulations and generalizations
  role of Wigner-Yanase-Dyson entropy in proof of SSA
- Simple proof of joint convexity of relative entropy
- Isaac Kim's operator version of SSA (2012)
- New appreciation (brief) Equality conditions
- New developments, open questions and challenges (brief)

Strong Subadditivity of Quantum Entropy

Def: (1927) von Neuman Entropy of quantum state ρ

density matrix $\rho \geq 0, \quad \text{Tr}\, \rho = 1$

$$S(\rho) = -\text{Tr}\, \rho \log \rho = -\sum_k \lambda_k \log \lambda_k$$

Props: 1) $S(\rho) \geq 0$ 2) $S(\rho)$ concave

3) SSA for multi-party systems $\mathcal{H} = \mathcal{H}_A \otimes \mathcal{H}_B \otimes \mathcal{H}_C$

$$S(\rho_{ABC}) + S(\rho_B) \leq S(\rho_{AC}) + S(\rho_{BC})$$

Conj. by Lanford, Robinson, Ruelle $\approx$ 1968; Lieb & Ruskai proved 1973

Delbrück and Moliere (1936) proved all props of $S(\rho)$ except SSA

One Interpretation of SSA

Subaddditivity $S(\rho_{J \cup K}) \leq S(\rho_J) + S(\rho_K)$

with equality $\iff$ $\rho_{J \cup K} = \rho_J \otimes \rho_K$, i.e., no interaction

Still true if they overlap – but count edintersection twice

$$S(\rho_{J \cup K}) + S(\rho_{J \cap K}) \leq S(\rho_J) + S(\rho_K)$$

Rewrite Using Relative Entropy

Relative entropy: $H(P,Q) = \text{Tr}\, P(\log P - \log Q) \qquad P,Q > 0$

Usually $\text{Tr}\, P = \text{Tr}\, Q$ which $\Rightarrow H(P,Q) \geq 0$; only need $P,Q > 0$

MPT: $H(P_A, Q_A) \leq H(P_{AB}, Q_{AB})$

$H(P,Q)$ decreases monotonically under partial trace

$$\begin{aligned} H(\rho_{BC}, \rho_C) &= \text{Tr}_{BC}\, \rho_{BC} \log \rho_{BC} - \text{Tr}_{BC}\, \rho_{BC} \log \rho_B \\ &= -S(\rho_{BC}) + S(\rho_B) \end{aligned}$$

$$\begin{aligned} H(\rho_{BC}, \rho_C) &\leq H(\rho_{ABC}, \rho_{AC}) \\ -S(\rho_{BC}) + S(\rho_B) &\leq -S(\rho_{ABC}) + S(\rho_{AB}) \end{aligned}$$

SSA written as ineq for conditional info $\equiv S(\rho_{BC}) - S(\rho_B)$

Equivalent Props of Relative Entropy

JC: $H(P, Q)$ jointly convex in P, Q

MPT: $H(P_A, Q_A) \leq H(P_{AB}, Q_{AB})$ monotone under partial trace

MQC $H[\Phi(P), \Phi(Q)] \leq H(P, Q)$ decreases under quantum channel

Φ models noise i.e. completely positive trace-preserving (CPT) map

Imply and equivalent to seemingly slightly weaker statements

Cond Info $S(\rho_{AB}) - S(\rho_B)$ concave in ρ_{AB}

SSA: $S(\rho_{ABC}) + S(\rho_B) \leq S(\rho_{AC}) + S(\rho_{BC})$

Focus: Prove joint convexity of $H(P, Q)$

Remarks on equivalence

JC $\Rightarrow$ MPT: based on observation of Uhlmann (1973)

indep and before proof of SSA rediscovered by Gromov 2012

- $\rho_B = \mathrm{Tr}_A \, \rho_{AB} = \int U_A \, \rho_{AB} \, U_A^* \, dU_A$ Haar measure on unitary
- pedestrian form $\rho_B = \mathrm{Tr}_A \, \rho_{AB} = \frac{1}{d_A} \sum_{jk} X^j Z^k \rho_{AB} \, Z^{-k} X^{-j}$

  X, Z Weyl-Heisenberg or gen Pauli ops, i.e., shift and phase on $\mathcal{H}_A$

MPT $\Rightarrow$ MQC: (Lindblad, 1974) based on Stinespring rep

$\Phi(\rho_A) = \mathrm{Tr}_E V \rho_A V^*$ $\quad V : \mathcal{H}_A \mapsto \mathcal{H}_B \otimes \mathcal{H}_E$ $\quad V^* V = I_A$

SSA $\Rightarrow$ Cond Info concave $\Rightarrow$ JC of $H(P, Q)$

(weak) subadd $\Rightarrow$ $S(\rho)$ concave by clever choice of block matrix

Other directions trivial

Conditional information

Cond Info $S(\rho_{AB}) - S(\rho_A) = -H(\rho_{AB}, \rho_A)$

concavity surprising because diff of concave functions

but now know special case of JC of $H(P, Q)$

Aside: cond info ≥ 0 for classical systems,

but neg for highly entangled quantum states !

once thought "defect"; now has nice info theory interp.

cond info is amount of info transfer need to learn AB knowing A

when neg, measures entanglement available for future info trans

M. Horodecki, Oppenheim and Winter (2005) state merging

Purification and Weak Monotonicity

Easy consequence of Singular Value Decomposition (Schmidt)

$\rho_{AD} = |\psi_{AD}\rangle\langle\psi_{AD}|$ pure $\Rightarrow$ $S(\rho_A) = S(\rho_D)$

For ρ_{ABC} can find pure $|\psi_{ABCD}\rangle$ s.t $\rho_{ABC} = \text{Tr}_D\, |\psi_{ABCD}\rangle\langle\psi_{ABCD}|$

Apply to SSA $S(\rho_{AB}) + S(\rho_{BC}) - S(\rho_{ABC}) - S(\rho_B) \geq 0$

Equiv. ineq: $S(\rho_{CD}) + S(\rho_{BC}) - S(\rho_D) - S(\rho_B) \geq 0$

Weak monotonicity (WM) or "monogamy of entanglement"

Cond Info $S(\rho_{BC}) - S(\rho_B)$ and $S(\rho_{CD}) - S(\rho_D)$ can't both be neg

Charlie can be entangled with Beverly or Dorothy, but not both

ρ_{ABD} pure $\Rightarrow$ LHS of WM $= 0$. But LHS sum of cond info
$\Rightarrow$ concave in ρ_{BCD} $\Rightarrow$ ≥ 0 First proof of SSA

Wigner-Yanase-Dyson entropy

skew entropy $\frac{1}{2}\text{Tr}\,[K,\gamma^t][K,\gamma^{1-t}] \;=\; \text{Tr}\,K\gamma^t K\gamma^{1-t} - \text{Tr}\,K\gamma K$

for $K = K^*$ observable and γ a density matrix

- Wigner-Yanase (1963) introduced $t = \frac{1}{2}$ proved concave in γ
- Dyson suggested $t \in (0,1)$ and $\lim t \to 0,1$

original WY paper and Lieb say $\lim\limits_{t\to 0,1}$ gives usual entropy for $K = I$

- But need to replace $\frac{1}{2}$ by $\frac{1}{t(1-t)}$ missing in WY. Only then

$\lim\limits_{t\to 1} \frac{1}{t(1-t)}\text{Tr}\,[K,\gamma^p][K,\gamma^{1-p}]\Big|_{K=I} = -\text{Tr}\,\gamma\log\gamma$ Dyson knew

WYD Conj: $\gamma \mapsto \text{Tr}\,K\gamma^t K\gamma^{1-t} - \text{Tr}\,K\gamma K$ concave

Lieb (1973) dropped linear term and proved generalization

$(P,Q) \mapsto \text{Tr}\,K^* P^t K\, Q^{1-t}$ concave for $t \in (0,1)$

Ando's proof and extension

Ando (1979) gave alternate proof that

- $\mathrm{Tr}\, K^* P^t K\, Q^{1-t}$ concave for $t \in (0,1)$
- also showed convex for $t \in (-1,0)$ and $t \in (1,2)$

Not until Hasegawa 1993; rediscovered (2010) Jencova - Ruskai

Include linear term and $\frac{1}{t(1-t)}$ get One Theorem for $t \in [-1,2]$

$$(P,Q) \mapsto \frac{1}{t(1-t)}\Big[\mathrm{Tr}\, K^* P K - \mathrm{Tr}\, K^* P^t K Q^{1-t}\Big] \text{ jointly convex}$$

Special cases: WYD entropy concave

and $H(P,Q) = \mathrm{Tr}\, P(\log P - \log Q)$ convex

Lindblad (1974) different approach $\lim_{t\to 1}$ Lieb's WYD $\Rightarrow$ JC of $H(P,Q)$

Lieb's long chain of inequalities brilliant, but not necessary

Araki's relative modular operator (1976)

Key development leading to extensions and simple proofs

Pedestrian version: $d \times d$ matrices Hilbert space $\langle P, Q \rangle = \operatorname{Tr} P^* Q$

Def. Left and Right mult as linear operators on this vector space

$$L_P(X) = PX \qquad \text{and} \qquad R_Q(X) = XQ$$

a) L_P and R_Q commute $\quad L_P[R_Q(X)] = PXQ = R_Q[L_P(X)]$

b) $P = P^* \quad \Rightarrow \quad L_P, R_P$ self-adjoint wrt H-S inner prod

For $P, Q > 0$ positive definite

c) L_P, R_P pos def $\quad \langle X, R_P(X) \rangle = \operatorname{Tr} X^* X P = \operatorname{Tr} X P X^* \geq 0$

d) $(L_P)^{-1} = L_{P^{-1}}, \quad (R_Q)^{-1} = R_{Q^{-1}}$

e) $f(L_P) = L_{f(P)}$, etc. $\quad \log R_Q = R_{\log Q}$

Define $H(P, Q)$ by relative modular operator

Formal $H(P, Q) = \text{Tr}\, P \log(P/Q)$ not well-defined, BUT

$$\begin{aligned} \text{Tr}\ \log(L_P R_Q^{-1})(P) &= \text{Tr}\ (\log L_P - \log R_Q)(P) \\ &= \text{Tr}\ \big(L_{\log P} - R_{\log Q}\big)(P) = \text{Tr}\, P\, (\log P - \log Q) \end{aligned}$$

$H(P, Q) \equiv \text{Tr}\,(\log \Delta_{PQ})(P) = \text{Tr}\, P\, (\log P - \log Q)$ well-def

$\Delta_{PQ} = L_P R_Q^{-1}$ called relative modular operator deep idea

Araki (1976) introduced to extend entropy and SSA

to type III von Neumann algebra which have no trace

In finite dims led to simple proofs of SSA and JC of $H(P, Q)$

and extensions of $H(P, Q)$ including WYD relative entropy

took MBR 25 years to realize this

Generalized relative entropy

Petz (and others) advocated relative modular proofs of SSA

Led Petz (1986) to generalize using term "quasi-entropy"

$$\mathcal{G} = \{g : (0,\infty) \mapsto \mathbf{R} \text{ operator convex}, \quad g(1) = 0\}$$

$$H_g(K,P,Q) \equiv \text{Tr}\,\sqrt{Q}K^*\, g_p\big(L_P R_Q^{-1}\big)(K\sqrt{Q})$$

$$g(x) \in \mathcal{G} \;\Leftrightarrow\; \widetilde{g}(x) = x\, g(x^{-1}) \in \mathcal{G}$$

$$\widetilde{H}_g(K,P,Q) = H_g(K^*,Q,P)$$

$$g(x) = x\log x \qquad H_g(I,P,Q) = \text{Tr}\,P\,(\log P - \log Q)$$

$$\widetilde{g}(x) = -\log x \qquad \widetilde{H}_g(I,P,Q) = \text{Tr}\,Q(\log Q - \log P)$$

Recover WYD Entropy

$$g_t(x) = \begin{cases} \frac{1}{t(1-t)}(x - x^t) & t \neq 1 \\ x \log x & t = 1 \end{cases}. \qquad t \in (0, 2]$$

$$\widetilde{g}_t(x) = x\, g_t(x^{-1}) = \begin{cases} \frac{1}{t(1-t)}(1 - x^t) & t \neq 0 \\ -\log x & t = 0 \end{cases} \qquad t \in [-1, 1)$$

$$\begin{aligned} J_t(K, P, Q) &\equiv \operatorname{Tr} \sqrt{Q} K^* g_t\big(L_P R_Q^{-1}\big)(K\sqrt{Q}) \qquad t \in [-1, 2] \\ &= \frac{1}{t(1-t)}\big(\operatorname{Tr} K^* P K - \operatorname{Tr} K^* P^t K Q^{1-t}\big) \end{aligned}$$

$$\begin{aligned} J_1(K, P, Q) &= \operatorname{Tr} K K^* P \log P - \operatorname{Tr} K^* P K \log Q \\ \widetilde{J}_0(K, P, Q) &= \operatorname{Tr} K^* K\, Q \log Q - \operatorname{Tr} K Q K^* \log P \end{aligned}$$

Recover both WYD entropy with linear term and $H(P, Q)$, $\quad K = I$

Integral representation

convex op

$$g(x) = ax + \int_0^\infty \frac{x^2 u - x}{x+u}\, d\nu(u)$$
$$= \int_0^\infty \Big[\frac{x^2}{x+u} - \frac{1}{u} + \frac{1}{x+u}\Big]\, u\, d\nu(u)$$

$$H_g(K,P,Q) = a\mathrm{Tr}\, K^* P K - \int_0^\infty \mathrm{Tr}\, K\, Q\, K^*\, d\nu(u)$$
$$+ \int_0^\infty \Big[\mathrm{Tr}\, K^* P \frac{1}{L_P + uR_Q}(PK) + \mathrm{Tr}\, QK^* \frac{1}{L_A + uR_Q}(KQ)\Big]\, u\, d\nu(u)$$

To show $H_g(K,P,Q)$ jointly convex in P,Q suffices

to show $(X,P,Q) \mapsto \mathrm{Tr}\, X^* \frac{1}{L_Q + tR_P}(X)$ jointly convex

Proof of Joint convexity of $X, P, Q \mapsto$ Tr

Note: $\operatorname{Tr}(\lambda X)^* \dfrac{1}{L_{\lambda Q} + uR_{\lambda P}}(\lambda X) = \lambda \operatorname{Tr} X^* \dfrac{1}{L_Q + uR_P}(X)$

Homo of degree 1 $\Rightarrow$ suffices to prove $F\Big(\sum_j \quad\Big) \geq \sum_j F(\ )_j$

Let: $M = (\ )^{-1/2}(X) - (\ )^{1/2}(\Lambda)$

$$\begin{aligned}
\operatorname{Tr} M^* M &= \langle M, M\rangle \\
&= \langle \big[(\ )^{-1/2}(X) - (\ )^{1/2}(\Lambda)\big], \big[(\ )^{-1/2}(X) - (\ )^{1/2}(\Lambda)\big]\rangle \\
&= \langle X, (\ )^{-1}(X)\rangle - \langle X, \Lambda\rangle - \langle \Lambda, X\rangle + \langle \Lambda, (\ )(\Lambda)\rangle
\end{aligned}$$

Choose $M(u) = (L_P + uR_Q)^{-1/2}(X) - (L_P + uR_Q)^{1/2}(\Lambda)$

$$\operatorname{Tr} M^*(u)M(u) =$$
$$\operatorname{Tr} X^*(L_P + uR_Q)^{-1}(X) - \operatorname{Tr} X^*\Lambda - \operatorname{Tr} \Lambda^* X + \operatorname{Tr} \Lambda^*(L_P + uR_Q)(\Lambda)$$

cont.

Let $M_j(u) = (L_{P_j} + uR_{Q_j})^{-1/2}(X_j) - (L_{P_j} + uR_{Q_j})^{1/2}(\Lambda)$. Then

$$0 \le \sum_j \operatorname{Tr} M_j^*(u) M_j(u) = \sum_j \operatorname{Tr} X_j^* (L_{P_j} + uR_{Q_j})^{-1}(X_j)$$
$$- \operatorname{Tr} \big(\textstyle\sum_j X_j^*\big)\Lambda - \operatorname{Tr} \Lambda^* \big(\textstyle\sum_j X_j\big) + \operatorname{Tr} \Lambda^* \textstyle\sum_j \big(L_{P_j} + uR_{Q_j}\big)\Lambda$$

Choose $\Lambda = \dfrac{1}{L_{\sum_j P_j} + uR_{\sum_j Q_j}} \big(\sum_j X_j\big)$. Use $\sum_j L_{P_j} = L_{\sum_j P_j}$

$$\operatorname{Tr} \big(\textstyle\sum_j X_j^*\big)\Lambda = \operatorname{Tr} \Lambda^* \big(\textstyle\sum_j X_j\big) = \operatorname{Tr} \big(\textstyle\sum_j X_j^*\big) \frac{1}{L_{\sum_j P_j} + uR_{\sum_j Q_j}} \big(\textstyle\sum_j X_j\big)$$
$$= \operatorname{Tr} \Lambda^* \textstyle\sum_j \big(L_{P_j} + tR_{Q_j}\big)\Lambda$$

$$0 \le \sum_j \operatorname{Tr} X_j^* \frac{1}{L_{P_j} + uR_{Q_j}}(X_j) - \operatorname{Tr} \big(\textstyle\sum_j X_j^*\big) \frac{1}{L_{\sum_j P_j} + uR_{\sum_j Q_j}} \big(\textstyle\sum_j X_j\big)$$

Aside on Schwarz inequality

Recall elementary inequality for numbers

$$\Big|\sum_k \overline{a}_k b_k\Big|^2 \;\le\; \sum_k |a_k|^2 \;\sum_k |b_k|^2$$

For $p_k > 0$ let $a_k = p_k^{1/2}, \quad b_k = p_k^{-1/2} x_k$

$$\Big|\sum_k x_k\Big|^2 \;\le\; \sum_k p_k \;\sum_k \overline{x}_k \frac{1}{p_k} x_k$$

Rewrite $\quad \Big(\sum_k \overline{x}_k\Big) \dfrac{1}{\sum_k p_k} \Big(\sum_k x_k\Big) \;\le\; \sum_k \overline{x}_k \dfrac{1}{p_k} x_k$

Compare proof: $\quad \sum_k \overline{m}_k m_k \;=\; \Big|\sum_k a_k + \lambda b_k\Big|^2 \ge 0 \quad \forall \;\; \lambda$

choose λ to minimize $\qquad$ same proof – choose operator Λ

Operator Schwarz ineuqlaity

$$\Big(\sum_k X_k^*\Big)\frac{1}{\sum_k P_k}\Big(\sum_k X_k\Big) \;\leq\; \sum_k X_k^* \frac{1}{P_k} X_k$$

Lieb and Ruskai proved (71-72 before SSA – proved special cases)

Not suff. for SSA — need Araki rel mod op hidden in L_P and R_Q.

Published 1974 – later papers with other proofs, but no earlier refs

summer, 2010 got paper to referee:

.... joint convexity of $X^*P^{-1}X$ proved by Kiefer in 1959, and rediscovered by Lieb and Ruskai ...

JSTOR J. Kiefer, "Optimum experimental designs", *J. Roy. Statist. Soc. Ser. B* **21** 272–310 (1959).

Kim's operator inequality

Isaac Kim (grad student Caltech) in *J. Math. Phys.*

arxiv:1210.5190 posted almost exactly 40 years after proof of SSA!

$$\mathrm{Tr}_{AB}\left[\log\rho_{ABC} - \log\rho_{AB} - \log\rho_{BC} + \log\rho_B\right]\rho_{ABC} \;\geq\; 0$$

N.B. only trace over $\mathcal{H}_A \otimes \mathcal{H}_B$. Have operator ineq on $\mathcal{H}_C$

Don't get new proof of SSA by taking trace over $\mathcal{H}_C$.

Revisit proofs using $\Delta_{PQ} = L_P R_Q^{-1}$ Rel Mod Op

Kim's operator inequality

Recall joint convexity $\Rightarrow$ MPT

$$H_g(I_A \otimes K_{BC}, P_{ABC}, Q_{ABC}) \quad \geq \quad H_g(K_{BC}, P_{BC}, Q_{BC})$$

Rewrite using def

$$\begin{aligned} \mathrm{Tr}_{ABC}\, I_A \otimes K_{BC}^*\, g\big(L_{P_{ABC}} R_{Q_{ABC}}^{-1}\big) K_{BC}\, Q_{ABC} & \\ \geq \quad & \mathrm{Tr}_{BC}\, K_{BC}^*\, g\big(L_{P_{BC}} R_{Q_{BC}}^{-1}\big) K_{BC}\, Q_{BC} \\ = \quad & \mathrm{Tr}_{ABC}\, K_{BC}^*\, g\big(L_{P_{BC}} R_{Q_{BC}}^{-1}\big) K_{BC}\, Q_{ABC} \end{aligned}$$

choose $K_{BC} = I_B \otimes K_C$ and $P_{ABC} = P_{AB} \otimes I_C$ which commute

$$\Rightarrow \quad K_C^*\, g\big(L_{P_{AB}} R_{Q_{ABC}}^{-1}\big) K_C\, Q_{ABC} = K_C^* K_C\, g\big(L_{P_{AB}} R_{Q_{ABC}}^{-1}\big)\, Q_{ABC}$$

Kim's operator inequality

$$\mathrm{Tr}_{ABC}\, I_{AB} \otimes K_C^* K_C\, g\big(L_{P_{AB}} R_{Q_{ABC}}^{-1}\big)\, Q_{ABC}$$
$$\geq \quad \mathrm{Tr}_{ABC}\, I_{AB} \otimes K_C^* K_C\, g\big(L_{P_B} R_{Q_{BC}}^{-1}\big)\, Q_{ABC}$$

Now choose $K_C = |\phi_C\rangle\langle\phi_C|$

$$\langle \phi_C, \mathrm{Tr}_{AB}\Big[g\big(L_{P_{AB}} R_{Q_{ABC}}^{-1}\big) - g\big(L_{P_B} R_{Q_{BC}}^{-1}\big)\Big]\, Q_{ABC}\ \phi_C\rangle \ \geq\ 0$$

$\forall \ |\phi_C\rangle \quad \Rightarrow \quad$ positive operator on $\mathcal{H}_C$

$$\mathrm{Tr}_{AB}\, g\big(L_{P_{AB}} R_{Q_{ABC}}^{-1}\big)\, Q_{ABC} - \mathrm{Tr}_B\, g\big(L_{P_B} R_{Q_{BC}}^{-1}\big)\, Q_{BC} \ \geq\ 0$$

SSA and reverse operator inequality

Apply to $g(x) = -\log x$

$$\mathrm{Tr}_{AB}\left[\log\rho_{ABC} - \log\rho_{AB} - \log\rho_{BC} + \log\rho_B\right]\rho_{ABC} \;\geq\; 0$$

Use $\widetilde{g}(x) = x\log x$ and $P \leftrightarrow Q$ get adjoint

$$\mathrm{Tr}_{AB}\,\rho_{ABC}\left[\log\rho_{ABC} - \log\rho_{AB} + \log\rho_B - \log_{BC}\right] \;\geq\; 0$$

Use $\widetilde{g}(x) = x\log x$ with same choice for P, Q get new ineq.

$$\mathrm{Tr}_{AB}\,\rho_{AB}\left[\log\rho_{AB} - \log\rho_{ABC} - \log\rho_B + \log\rho_{BC}\right] \;\geq\; 0$$

looks like "reverse SSA" since inside $[\qquad]$ has opposite sign

Question: Can pair of new ineq be used for upper and lower bounds in some applications??

WYD operator inequality

Using $g(x) = \frac{1}{t(1-t)}(1 - x^t)$ or $\widetilde{g}(x) = \frac{1}{t(1-t)}(x - x^{1-t})$

$$\frac{1}{t(1-t)}\left[\mathrm{Tr}_{AB}\, \rho_{ABC}^{1-t}\, \gamma_{AB}^{t} - \mathrm{Tr}_{B}\, \rho_{BC}^{1-t}\ \gamma_{B}^{t}\right] \geq 0 \qquad \forall \ t \in [-1, 2]$$

Need linear term to get joint concavity $\Rightarrow$ MPT

but linear terms cancels in MPT

New appreciation

1973 MBR gave talk at IUPAP and session chair asked

"beautiful theorem, but what use is it"

worked on other things in Schrödinger operators for $\approx$ 20 years

$\approx$ 1997 starting getting e-mail about SSA (sabbat GaTech)

Shor's algorithm factoring large numbers on QC led to interest in
Quantum Computing and Qunatum Information Theory

- many applications of SSA in Quantum Info
- now in all textbooks – even early notes Preskill

One question: What are equality conditions?

Equality conditions

Recall SSA $S(\rho_{ABC}) + S(\rho_B) \leq S(\rho_{AC}) + S(\rho_{BC})$

$\rho_{ABC} = \rho_{AB} \otimes \rho_C$ or $\rho_{ABC} = \rho_A \otimes \rho_{BC}$ suffice but not nec

For nasc for equality go back to proof of joint convexity

$$M_j(u)M_j(u) = 0 \ \ \forall \ \ j, \qquad \forall \ \ u \ \in (0, \infty)$$

Write out and translate from JC to MPT.

Omit details – key point $\forall \ \ u \ \in (0, \infty)$

$\Rightarrow$ equality conds indep of $g(x)$ if ν_g supported on $(0, \infty)$

$$\mathcal{H}_B = \bigoplus_k \mathcal{H}_{B'_k} \otimes \mathcal{H}_{B''_k} \qquad\qquad \rho_{ABC} = \bigoplus_k \rho^k_{AB'} \otimes \rho^k_{B''C}$$

Form worked out by Hayden, Jozsa, Petz, Winter (2004)

equality conds same for Kim operator inequality forms

Why Important in quantum information

- Holevo bound on accessible info in quantum ensemble

  1973 long and complicated proof (indep of SSA)

  Holevo bound also not appreciated for $>$ 20 years

But can now prove as an easy corollary of SSA

- applications in channel capacity

N. Linden and A. Winter, *Commun. Math. Phys.* **259**, 129–138 (2005).
"A New Inequality for the von Neumann Entropy" quant-ph/0406162

130 N. Linden, A. Winter

most importantly, there are *inequalities* governing the relative magnitude of entropies, conditional entropies and mutual information.

In the quantum case, there is essentially only one known inequality (all others being derivable from it): strong subadditivity. Proved by Lieb and Ruskai [6] in 1973, it is the key result on which virtually every nontrivial quantum coding theorem relies.

We prove here a new inequality for the von Neumann entropy, which we show cannot be derived from the known ones: it is a *constrained inequality* in that it is not true in general but only for states satisfying three particular linear constraints on their entropies.

Another new convexity inequality

Lieb proved $P \mapsto \operatorname{Tr} e^{K+\log P}$ concave.

- Played important role in original proof of SSA
- Uhlmann (1973) indep realized that this would imply SSA
- Epstein (1973) simple proof $f(x) = \operatorname{Tr} e^{K+\log(P+xQ)}$

  showed $z\, f(z^{-1})$ Herglotz, i.e., maps UHP $\mapsto$ UHP

Aside: first meeting with Dyson   Epstein must be read backwards

New result: Harvey and Olver (2012) $P_k > 0, Q_k \geq 0.$

$$x \mapsto \operatorname{Tr} exp\big[K + \log(P_1 + x\, Q_1) + \log(P_2 - x \log Q_2)\big]$$

is concave in x near $x = 0$.

- Proved using Epstein's method ($\pm$ signs on Q_k important)

used to study paving problem version of Kadison-Singer conjecture

Future directions

- SSA looks like first in chain of n-party inequalities
- Lieb (1975) showed natural 4-party generalizations false

BUT recent discovery of non-Shannon classical entropy inequalities suggests there are non-obvious new quantum inequalities

1997-8 Yeung-Zhang found 4-party classical entropy ineq indep of

(a) $S(p) \geq 0$ (b) $S(p_{AB}) \geq S(p_A)$ (c) SSA

2005 DFZ more by computer algebra, 2007 Matúš found infinite families

Extensions to quantum systems hard open question

Proofs do not extend because "no-cloning" of quantum states

But can say something about inequalities for stabilizer states

Ingleton inequality

mutual information

$$I(A:B) \equiv S(\rho_A) + S\rho_(B) - S\rho_{(AB})$$

conditional mutual information

$$I(A:B|C) \equiv S(\rho_{AC}) + S(\rho_{BC}) - S(\rho_C) - S(\rho_{ABC})$$

Ingleton expression

$$\mathrm{ING}(AB:CD) \equiv I(A:B|C) + I(A:B|D) + I(C:D) - I(A:B)$$

SSA equiv to $I(A:B|C) \geq 0$

Ingleton inequality $\mathrm{ING}(AB:CD) \geq 0$

not universal – simplest "linear rank inequality"

Examples of "balanced" inequality – number of $A, B, \ldots$ cancel out

Matúš family of 4-party non-Shannon inequalities

Gives two infinite families, $m = 1, , 2, 3 \ldots$ positive integer

$$m\,\mathrm{ING}(AB:CD) + I(A:B|D) + \tfrac{m(m+1)}{2}\big[I(B:D|C) + I(C:D|B)\big] \;\geq\; 0$$

$\mathrm{ING}(AB:CD) +$ positive terms $\quad \geq 0$

$\Rightarrow \quad$ 4-party entropy cone not polyhedral

suggests don't yet know all classical 4-party inequalities

Chan and Yeung (2002)

showed 1-1 relation between entropy and group rank ineq.

SSA : $\quad |G_1| \cdot |G_2| \leq |G_1 \cap G_2| \cdot |G|$

Entropy of stabilizer states

Important in quantum information:

- stabilizer subspaces first quantum error correction codes
- simul eigenstates of max Abel subgroup of Weyl-Heis group

Thm: (2012, Linden, Matúš, Ruskai, Winter)

$\rho_{ABCD} = |\psi_{ABCDE}\rangle\langle\psi_{ABCDE}|$ with $|\psi_{ABCDE}\rangle$ stabilizer state

$\iff$ positivity, SSA and Ingleton hold roughly true

Thm: Stabilizer states satisfy all linear rank (vector space subspace) inequalities obtained from "common information".

Thm: (indep by Gross and Walter) Stabilizer states satisfy all (balanced) classical entropy inequalities.

Open questions

- Find example of "true" quant state which violates Ingleton

- Are all quantum entropy vectors which violate Ingleton in classical entropy cone??

- Do new classical entropy ineq extend to quantum systems?

- What inequalities characterize quantum entropy cone?

- All class ineq of form linear rank ineq + pos terms ≥ 0?

- Do stabilizer states satisfy linear rank inequalities that do not arise from common info ?

- Find explicit example of such an inequality. (Have exist Thm)

- How much of a restriction are new inequalities, i.e., relative size of true entropy cone and Shannon or vonNeuman cone

Conclusion and challenges

- Find and prove new “non-Shannon” or “non vonNeumann” inequalities for quantum entropy, i.e.,

  quantum inequalities stronger than strong subadditivity

- Need Better Parties

Compare photos of von Neumann and MBR with pineapple on head


JOHN VON NEUMANN