Contraction Coefficients for Noisy Quantum Channels

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Overview

- Review notation and definitions
  - First define using operator convex functions
    - a) Relative entropy and generalizations
    - b) Riemannnian metric or Fisher information
    - c) Geodesic distance
  - Basic property — decrease under quantum channels
  - Contraction coefficient: measures how much
- Old Results from Lesniewski/Ruskai
- New Results with F. Hiai

Operators on Matrices

Consider linear ops on M_d space of $d \times d$ matrices as Hilbert space

with Hilbert-Schmidt inner prod $\langle P, Q \rangle = \text{Tr}\, P^* Q$

denote adjoint of Φ by $\widehat{\Phi}$, i.e., $\text{Tr}\, [\Phi(P)]^* Q = \text{Tr}\, P^* \widehat{\Phi}(Q)$

Example: Quantum channel Φ

completely positive, trace preserving map on M_d, i.e.,

$\mathcal{I} \otimes \Phi$ positivity preserving ("positive")on $M_d \times M_d$

Density matrices $\mathcal{D} \equiv \{P \in M_d : P \geq 0, \text{Tr}\, P = 1\}$

Tangent space $\{A = M_d : A = A^*, \ \text{Tr}\, A = 0\}$

Basic Tool

Construct operators and functions of operators using

Def. Left and Right mult as linear operators on this vector space

$$L_P(X) = PX \qquad \text{and} \qquad R_Q(X) = XQ$$

a) L_P and R_Q commute $L_P[R_Q(X)] = PXQ = R_Q[L_P(X)]$

b) $P = P^* \;\Rightarrow\; L_P, R_P$ self-adjoint wrt H-S inner prod

For $P, Q > 0$ positive definite

c) L_P, R_P pos def $\langle X, R_P(X)\rangle = \text{Tr}\, X^*XP = \text{Tr}\, XPX^* \geq 0$

d) $(L_P)^{-1} = L_{P^{-1}}, \quad (R_Q)^{-1} = R_{Q^{-1}}$

e) $f(L_P) = L_{f(P)}$, etc. $\quad \log R_Q = R_{\log Q}$

Generalized Relative Entropy

Petz (1986) defined "quasi-entropy"

a.k.a. "generalized realtive entropy", "f-divergence"

$$\mathcal{G} = \{g : (0, \infty) \mapsto \mathbf{R} \mid \text{operator convex, } g(1) = 0\}$$

$$H_g(K, P, Q) \equiv \text{Tr} \sqrt{Q} K^* \, g_p \big(L_P R_Q^{-1}\big)(K \sqrt{Q})$$

Thm: $H_g(K, P, Q)$ jointly convex in P, Q $\Rightarrow$ monotonicity results

$g(x) \in \mathcal{G} \;\Leftrightarrow\; \widetilde{g}(x) = x\, g(x^{-1}) \in \mathcal{G}$

$$\widetilde{H}_g(K, P, Q) = H_g(K^*, Q, P)$$

$g(x) = x \log x$ $\qquad$ $H_g(I, P, Q) = \text{Tr}\, P\,(\log P - \log Q)$

$\widetilde{g}(x) = -\log x$ $\qquad$ $\widetilde{H}_g(I, P, Q) = \text{Tr}\, Q(\log Q - \log P)$

Recover WYD Entropy

$$g_t(x) = \begin{cases} \frac{1}{t(1-t)}(x - x^t) & t \neq 1 \\ x \log x & t = 1 \end{cases} . \qquad t \in (0, 2]$$

$$\widetilde{g}_t(x) = x\, g_t(x^{-1}) = \begin{cases} \frac{1}{t(1-t)}(1 - x^t) & t \neq 0 \\ -\log x & t = 0 \end{cases} \qquad t \in [-1, 1)$$

$$\begin{aligned} J_t(K, P, Q) &\equiv \operatorname{Tr} \sqrt{Q} K^* g_t\big(L_P R_Q^{-1}\big)(K\sqrt{Q}) \qquad t \in [-1, 2] \\ &= \frac{1}{t(1-t)}\big(\operatorname{Tr} K^* P K - \operatorname{Tr} K^* P^t K Q^{1-t}\big) \end{aligned}$$

$$\begin{aligned} J_1(K, P, Q) &= \operatorname{Tr} K K^* P \log P - \operatorname{Tr} K^* P K \log Q \\ \widetilde{J}_0(K, P, Q) &= \operatorname{Tr} K^* K\, Q \log Q - \operatorname{Tr} K Q K^* \log P \end{aligned}$$

Recover both WYD entropy with linear term and $H(P, Q)$, $K = I$

Riemannian metrics or Fisher information

$$-\frac{\partial^2}{\partial a \partial b} H_g(P + aA, P + bB, I)\Big|_{a=b=0} = \operatorname{Tr} A\, \Omega_P^k(B)$$
$$= \langle A, \Omega_P(B) \rangle \equiv \Gamma_P^k(A, B)$$

for $\operatorname{Tr} A = \operatorname{Tr} B = 0$ in LHS, get pos quad form which is RHS with

$$\Omega_P^k(X) \equiv R_P^{-1} k\left(L_P R_P^{-1}\right) X = L_P^{-1} k\left(R_P L_P^{-1}\right)$$

with $k(x) = \frac{g(x) + x g(x^{-1})}{(1-x)^2} = \frac{g_{\rm sym}(x)}{(1-x)^2} \in \mathcal{K}$ relate H_g and Γ_P^k

$\mathcal{K} = \{k : (0, \infty) \mapsto \mathbf{R} \mid k \text{ op convex }, k\left(x^{-1}\right) = x k(x)\}$

Ω_P non-commutative multiplication by P^{-1}

Examples

	$k(x)$	$\Omega_P(X)$
RelEnt	$\frac{\log x}{x-1}$	$\int_0^\infty \frac{1}{P+tI} X \frac{1}{P+tI}\, dt$
WYD	$\frac{1}{t(1-t)} \frac{(1-x^t)(1-x^{1-t})}{(1-x)^2}$ $t \in [-1, 2]$	$\frac{4}{\left(\sqrt{L_P}+\sqrt{R_P}\right)^2}$ $t = 1/2$
	$\frac{1}{2}\left(x^{-t} + x^{-1+t}\right)$ $t \in [0, 1]$	$\frac{1}{2}\left(P^{-t} X P^{-1+t} + P^{-1+t} X P^{-t}\right)$ $P^{-1/2} X P^{-1/2} \qquad t = 1/2$
max	$\frac{1+x}{2x}$	$\frac{1}{2}\left(XP^{-1} + P^{-1}X\right)$
min	$\frac{2}{1+x}$	$\frac{2}{L_P+R_P}(X)$

Geodesic distance

Define geodesic distance for each $k \in K$

$$D_k(P,Q) \equiv \inf_{\xi(t)} \int_0^1 \sqrt{\mathrm{Tr}\,\xi'(t)\,\Omega_{\xi(t)}\xi'(t)}dt$$
$$= \inf_{\xi(t)} \int_0^1 \sqrt{\Gamma_{\xi(t)}\Big(\xi'(t)\xi'(t)\Big)}dt$$

where $x(t)$ smooth path with $\xi(0) = P,\ \xi(1) = Q$

Know explicitly only in one case, Bures metric

Don't know if matrix metric $\|\log P - \log Q\|$ in this framework ?

$$\log P - \log Q = \int_0^\infty \frac{1}{Q + xI}(P - Q)\frac{1}{P + xI}dx$$

Bures metric and trace distance

Know geodesic distance explicitly only for min $k(x) = \frac{2}{1+x}$
$\Omega_P(X) = \frac{1}{L_P + R_P}(X)$ Bures metric studied by Uhlmann

$$\begin{aligned} D_k(P,Q) &= \inf_{Y,Z} \{ \text{Tr}\,(Y-Z)^*(Y-Z) : Y^*Y = P,\ Z^*Z = Q \} \\ &= 2\Big[1 - \text{Tr}\left(\sqrt{P}Q\sqrt{P}\right)^{1/2}\Big] = 2\,[1 - P\#Q] \end{aligned}$$

$P\#Q$ known as "fidelity" in quantum info

will also use trace distance for which formally

$$\text{Tr}\,|P-Q| = H_g(P,Q)$$

with $g(x) = |x-1|$ (obviously not op convex since not diff.)

Contraction Theorems

All of above decrease under quantum channels, i.e.,

for any CPT map Φ and for all $P, Q \in \mathcal{D}$ and $\text{Tr}\, A = 0$

Thm: $H_g[\Phi(P), \Phi(Q)] \leq H_g(P, Q) \qquad \forall\, g \in \mathcal{G}$

Thm: $\text{Tr}\, \Phi(A)\, \Omega^k_{\Phi(P)} \Phi(A) \leq \text{Tr}\, A\, \Omega^k_P(A) \qquad \forall\, k \in \mathcal{K}$

Thm: $D_k[\Phi(P), \Phi(Q)] \leq D_k(P, Q) \qquad \forall\, k \in \mathcal{K}$

Thm: $|\text{Tr}\, \Phi(P) - \Phi(Q)| \leq \text{Tr}\, |P - Q|$

$k(x) = \frac{g(x) + x g(x^{-1})}{(1-x)^2} = \frac{g_{\rm sym}(x)}{(1-x)^2}$ op conv and $k(x^{-1}) = x k(x)$

Definition of Contraction Coefficients

$$\eta_g(\Phi)^{\rm RelEnt} \equiv \sup_{P,Q} \frac{H_g[\Phi(P),\Phi(Q)]}{H_g(P,Q)}$$

$$\eta_k(\Phi)^{\rm Riem} \equiv \sup_{P\in\mathcal{D},\,\mathrm{Tr}\,A=0} \frac{\mathrm{Tr}\,\Phi(A)\,\Omega^k_{\Phi(P)}\Phi(A)}{\mathrm{Tr}\,A\,\Omega^k_P(A)}$$

$$\eta_k(\Phi)^{\rm geod} \equiv \sup_{P,Q} \frac{D_k[\Phi(P),\Phi(Q)]}{D_k(P,Q)}$$

$$\eta^{\rm Dob}(\Phi) = \eta^{\rm Tr}(\Phi) \equiv \sup_{P,Q} \frac{|\mathrm{Tr}\,\Phi(P)-\Phi(Q)|}{\mathrm{Tr}\,|P-Q|}$$

$\eta^{\rm Tr}$ quant gen of class Dobrushin coefficient of ergodicity

Old easy results (Lesniewski-Ruskai)

Follow easily from defs with $k(x) = \frac{g_{\rm sym}}{(x-1)^2}$

a) $$\eta_k(\Phi)^{\rm geod} \leq \eta_k(\Phi)^{\rm Riem} \leq \eta_{g_{\rm sym}}^{\rm RelEnt}(\Phi) \leq \eta_g^{\rm RelEnt}(\Phi) \leq 1$$

b) Can also show $$\eta_k^{\rm Riem}(\Phi) \geq \sqrt{\eta^{\rm Tr}(\Phi)}$$

c) For unital qubit channels $\Phi_T : [I + \mathbf{w} \cdot \sigma] \mapsto I + (T\mathbf{w}) \cdot \sigma$

$$\eta_k(\Phi)^{\rm geod} = \eta_k(\Phi)^{\rm Riem} = \eta_g^{\rm RelEnt}(\Phi) \leq 1 = \|T\|^2 \qquad \forall\ k, g$$

d) For non-unital CQ qubit channels $\eta_k(\Phi)^{\rm Riem}$ depends on k

Last two stated without proof in Lesniewski-Ruskai

Reformulation of $\eta_k(\Phi)^{\rm Riem}$ as eigenvalue problem

Use HS inner product $\langle X, Y \rangle = \text{Tr}\, X^* Y$ and $\widehat{\Phi}$ denote adjoint

$$\widehat{\Phi} \circ \Omega^k_{\Phi(P)} \circ \Phi(X) = \lambda \Omega^k_P(X)$$

equiv. to

$$\big[(\Omega_P)^{-1} \circ \widehat{\Phi} \circ \Omega^k_{\Phi(P)}\big] \Phi(X) = \lambda X$$

By max-min principle

$$\lambda_2(\Phi, P) = \sup_{\text{Tr}\, A = 0} \frac{\langle \Phi(A)\, \Omega^k_{\Phi(P)} \Phi(A) \rangle}{\langle A\, \Omega^k_P(A) \rangle}$$

$$\eta_k^{\rm Riem}(\Phi) = \sup_P \lambda_2(\Phi, P) \quad = \sup_{P \in \mathcal{D}, \text{Tr}\, A = 0} \frac{\langle \Phi(A)\, \Omega^k_{\Phi(P)} \Phi(A) \rangle}{\langle A\, \Omega^k_P(A) \rangle}$$

Recent results and conjectures

Can show $(\Omega_P)^{-1} \circ \widehat{\Phi} \circ \Omega^k_{\Phi(P)}$ pos pres $\Rightarrow$ $\eta_k^{\mathrm{Riem}}(\Phi) \le \eta^{\mathrm{Tr}}(\Phi)$

Cor: $\eta^{\mathrm{Riem}}_{k(x)=x^{-1/2}}(\Phi) \le \eta^{\mathrm{Tr}}(\Phi)$

Only $k(x)$ for which both Ω_P and $(\Omega_P)^{-1}$ are C.P. is $x^{-1/2}$

plays important role in quantum Markov processes

Conj: (Kastoryano-Temme) $\eta_k^{\mathrm{Riem}}(\Phi) \le \eta^{\mathrm{Riem}}_{k(x)=x^{-1/2}}(\Phi) \quad \forall\, k \in \mathcal{K}$

Conj: (Ruskai) $\eta_k^{\mathrm{Riem}}(\Phi) \le \eta^{\mathrm{Tr}}(\Phi) \qquad \forall\, k \in \mathcal{K}$

will see both are false

Aside: more on $\mathcal{K}$ and Ω_P

Recall $\mathcal{K} = \{k : (0,\infty) \mapsto \mathbf{R} \mid k \text{ op convex }, k(x^{-1}) = xk(x)\}$

Can verify $\qquad k \in \mathcal{K} \Leftrightarrow \widetilde{k}(x) \equiv 1/k(x^{-1}) \in \mathcal{K}$

$\mathcal{K}$ is a convex set with extreme points $k_\nu = \frac{1+x}{(x+\nu)(1+x\nu)} \frac{(1+\nu)^2}{2}$

$$k(x) = \int_0^1 \left(\frac{1}{x+\nu} + \frac{1}{1+x\nu} \right) \frac{1+\nu}{2} dm(\nu) = \int_0^1 \frac{1+x}{(x+\nu)(1+x\nu)} \frac{(1+\nu)^2}{2} dm(\nu)$$

$$\Omega_P^k(X) \equiv R_P^{-1} k \left(L_P R_P^{-1} \right) X = L_P^{-1} k \left(R_P L_P^{-1} \right)$$

Petz uses $f = 1/k$ with different conventions but equivalent result

Ω_P non-commutative multiplication by P^{-1}

$(\Omega_P)^{-1} \neq \Omega_{P^{-1}}$ non-commutative multiplication by P

Aside on CP of Ω_P and $(\Omega_P)^{-1}$

Example: $$\Omega_P^{\log}(X) = \int_0^\infty \frac{1}{P+tI} X \frac{1}{P+tI}\, dt$$

$$\left(\Omega_P^{\log}\right)^{-1}(Y) = \int_0^\infty P^u \, Y \, P^{1-u} du$$

$$\mathcal{K} = \{k : (0,\infty) \mapsto \mathbf{R} \mid k \text{ op convex }, k\left(x^{-1}\right) = xk(x)\}$$

$$\mathcal{K}^+ = \{k \in \mathcal{K} : \Omega_P^k \text{ is C.P. } \forall \; P \in \mathcal{D}\}$$

$$\mathcal{K}^- = \{k \in \mathcal{K} : \left(\Omega_P^k\right)^{-1} \text{ is C.P. } \forall \; P \in \mathcal{D}\}$$

$$k(x) \in \mathcal{K}^+ \quad \Leftrightarrow \quad \widetilde{k}(x) = 1/k(x^{-1}) \in \mathcal{K}^-$$

$$\mathcal{K}^+ \cap \mathcal{K}^- = \{k(x) = x^{-1/2}\}$$

WYD $k_t(x) \quad \in \mathcal{K}^+ \;\; t \in [0,1], \qquad \in \mathcal{K}^- \;\; t \in [-1, -\frac{1}{2}] \cup (\frac{3}{2}, 2]$

$\preccurlyeq$ order and C.P. conds

Def: $k_1 \preccurlyeq k_2$ if $k_1(e^t)/k_2(e^t)$ is pos def in Bochner sense

Fourier transform is positive

Equiv cond: matrix with els $\frac{k_1(x_j/x_k)}{k_2(x_j/x_k}$ is pos semi-def

Thm: TFAE

a) $k \in \mathcal{K}^+$, i.e., Ω_P is C.P.

b) $k \preccurlyeq x^{-1/2}$

c) $F(t) = e^t k(e^{2t})$ is pos def

Get above results and analyze many families using these conds

Summary

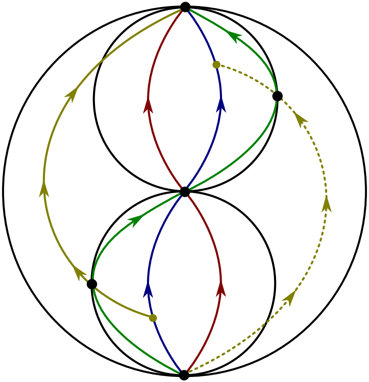

Figure: Diagram of families in $\mathcal{K}$ parameterized to increase in $\preccurlyeq$ order with the lower ball for $\mathcal{K}^+$ and the upper $\mathcal{K}^-$. The red curve describes the Heinz family $k_\alpha^{\rm H}$ $(0, \frac{1}{2})$ and $\widetilde{k}_\alpha^{\rm H}$ $(\frac{1}{2}, 1)$; the blue curve the binomial family $k_{-\alpha}^{\rm B}$ $(-1, 1)$; the green curve the power difference family $k_{-\alpha}^{\rm PD}$ $(-2, 1)$. The left brown curve $k_t^{\rm WYD}$ with $t \in [\frac{1}{2}, 2]$ and the right dotted brown curve the dual $\widetilde{k}_t^{\rm WYD}$. Note crossings at $\frac{4}{(1+\sqrt{x})^2}$ and $\frac{\log x}{x-1}$

Hiai's new results

Thm: (Hiai-Petz) $\langle A, \Omega_p^k(A)\rangle = \lim_{\epsilon\to 0} D_k(P, P+\epsilon A)$

Cor: $\eta_k^{\rm Riem}(\Phi) \le \eta_k^{\rm geod}(\Phi)$ $\Rightarrow$ Cor: $\eta_k^{\rm Riem}(\Phi) = \eta_k^{\rm geod}(\Phi)$

since opposite ineq elementary

Thm: (Hiai) $\eta_{\log}^{\rm RelEnt}(\Phi) = \eta_{\log}^{\rm Riem}(\Phi)$

Thm: (Hiai) $\eta_{\rm Bur}^{\rm Riem}(\Phi) = \eta_{\rm Bur}^{\rm RelEnt}(\Phi)$ $\quad k(x) = \frac{2}{1+x}$

Conj: $\eta_g^{\rm RelEnt}(\Phi) = \eta_k^{\rm Riem}(\Phi)$ $\quad \forall\ k(x) = \frac{g_{\rm sym}}{(x-1)^2}$

Unital qubit channels

$$\Phi : P = \tfrac{1}{2}[I + \mathbf{w}\cdot\sigma] \mapsto \tfrac{1}{2}\left[I + \textstyle\sum_k \alpha_k w_k \sigma_k\right]$$

$$\text{rep in Pauli basis } \begin{pmatrix} 1 & 0 \\ 0 & T \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \alpha_1 & 0 & 0 \\ 0 & 0 & \alpha_2 & 0 \\ 0 & 0 & 0 & \alpha_3 \end{pmatrix}$$

$$\eta_k(\Phi)^{\text{geod}} = \eta_k(\Phi)^{\text{Riem}} = \eta_g^{\text{RelEnt}}(\Phi) \leq 1 = \|T\|^2 = \max_k \alpha_k^2 \qquad \forall\, k, g$$

Proof exploits fact that for $\Gamma_{\mathbf{w}}$ pos lin op on $\mathbf{R_3}$

$$\sup_{\mathbf{y}\in\mathbf{R}^3} \frac{\langle \mathrm{T}\mathbf{y}, \Gamma_{\mathrm{T}\mathbf{w}}^{-1}\mathrm{T}\mathbf{y}\rangle}{\langle \mathbf{y}, \Gamma_{\mathbf{w}}^{-1}\mathbf{y}\rangle} = \sup_{\mathbf{y}\in\mathbf{R}^3} \frac{\langle \mathrm{T}^*\mathbf{y}, \Gamma_{\mathbf{w}}\mathrm{T}^*\mathbf{y}\rangle}{\langle \mathbf{y}, \Gamma_{\mathrm{T}\mathbf{w}}\mathbf{y}\rangle}.$$

CQ qubit channels

$$\Phi : P = \tfrac{1}{2}[I + \mathbf{w}\cdot\sigma] \mapsto \tfrac{1}{2}\big[I + \alpha w_1 \sigma_x + \tau \sigma_z\big] \qquad \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \alpha & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \tau & 0 & 0 & 0 \end{pmatrix}$$

CP cond $\qquad \alpha^2 + \tau^2 \leq 1$

extreme $k_\nu = \frac{1+x}{(x+\nu)(1+x\nu)} \frac{(1+\nu)^2}{2} \qquad \eta_{k_\nu}^{\rm Riem} = \frac{\alpha^2}{1 - \left(\frac{1-\nu}{1+\nu}^2\right)^2 \tau^2} \geq \alpha^2$

implies $\eta_k^{\rm Riem}(\Phi_{\alpha,\tau})$ depends on $k(x)$ and $\quad \geq \alpha^2 \;\forall\; k \in \mathcal{K}$

Also since $\eta_k^{\rm Riem}(\Phi) \geq \sqrt{\eta^{\rm Tr}(\Phi)}$ we have

$$\eta^{\rm Tr}(\Phi_{\alpha,\tau}) = \alpha \quad \Rightarrow \quad \eta_k^{\rm Riem}(\Phi_{\alpha,\tau}) \geq \alpha^2 \qquad \forall\; k \in \mathcal{K}$$

CQ (cont)

Can show by direct (usually tedious for $=$) computation

	harm	$\widetilde{\mathrm{WY}}$	geom	log	WY	Bur
$k(x)$	$\frac{1+x}{2x}$	$\frac{(1+\sqrt{x})^2}{4x}$	$x^{-1/2}$	$\frac{\log x}{x-1}$	$\frac{4}{(1+\sqrt{x})^2}$	$\frac{2}{1+x}$
	$=$	$\geq$	$\geq$	$\geq$	$=$	$=$
η_k^{Riem}	$\frac{\alpha^2}{1-\tau^2}$	$\frac{1+\sqrt{1-\tau^2}}{2(1-\tau^2)}$	$\frac{\alpha^2}{\sqrt{1-\tau^2}}$	$\frac{\alpha^2}{2\tau}\log\frac{1+\tau}{1-\tau}$	$\frac{2\,\alpha^2}{1+\sqrt{1-\tau^2}}$	α^2

Can verify both $k(x)$ and η_k^{Riem} in decreasing order above

Conj: Equality holds, at least when $\alpha^2 + \tau^2 = 1$ (not easier)

some numerical evidence against

Show some conjectures false

$k(x)$	$\frac{1+x}{2x}$	$\frac{(1+\sqrt{x})^2}{4x}$	$x^{-1/2}$	$\frac{\log x}{x-1}$	$\frac{4}{(1+\sqrt{x})^2}$	$\frac{2}{1+x}$
	$=$	$\geq$	$\geq$	$\geq$	$=$	$=$
$\eta_k^{\rm Riem}$	$\frac{\alpha^2}{1-\tau^2}$	$\frac{1+\sqrt{1-\tau^2}}{2(1-\tau^2)}$	$\frac{\alpha^2}{\sqrt{1-\tau^2}}$	$\frac{\alpha^2}{2\tau} \log \frac{1+\tau}{1-\tau}$	$\frac{2\,\alpha^2}{1+\sqrt{1-\tau^2}}$	α^2

- For $\frac{1+x}{2x}$ and $\alpha < 1 - \tau^2$, $\quad \eta^{\rm Riem} = \frac{\alpha^2}{1-\tau^2} > \alpha = \eta^{\rm Tr}$ for $\tau \neq 0$

  e.g., for $\alpha = \tau = 1/\sqrt{2}$, $\quad \eta_{\frac{1+x}{2x}}^{\rm Riem} = 1 > \alpha = \eta^{\rm Tr}$

- Also shows can have some $= 1$ and others < 1
- For $x^{-1/2}$ with $\alpha^2 + \tau^2 = 1$, $\eta^{\rm Riem} = \eta^{\rm Tr} = \alpha$ not largest

CQ (cont)

Further calculations show

- Can have $\eta_g^{\rm RelEnt} > \eta_k^{\rm Riem}$ for ext points.

Summary: In general with strict $<$ possible (probably generic)

$$\sqrt{\eta^{\rm Tr}(\Phi)} \leq \eta_k(\Phi)^{\rm geod} = \eta_k(\Phi)^{\rm Riem} \leq \eta_{g_{\rm sym}}^{\rm RelEnt}(\Phi) \leq \eta_g^{\rm RelEnt}(\Phi) \leq 1$$

Open questions

Still some open questions about when Ω_P^k C.P.

Equality in some $\eta_k^{\rm Riem}$ in bounds for CQ channel?

η for unital channels with $d > 2$??

η for random unitaries??

When does k_m inc p.w or in $\preccurlyeq$ order $\Rightarrow$ $\eta_{k_m}^{\rm Riem}(\Phi)$ increase?
or decrease? Hiai partial results for QC and CQ channels

When does some $\eta_{k_m}(\Phi) = 1 \;\Rightarrow\;$ others $= 1$ also ?
When does some $\eta_{k_m}(\Phi) < 1 \;\Rightarrow\;$ others < 1 also ?

How to number slides backwards in beamer??

References

D. Petz (1986) and (1996) and others given in refs below

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