Gour-Friedland proof (arXiv:1105.6122) of local additivity of minimal output entropy

Mary Beth Ruskai
marybeth.ruskai@tufts.edu

Tufts University and IQC Waterloo

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Background

Additivity Conj: $S_{\min}(\Phi_A \otimes \Phi_C) = S_{\min}(\Phi_A) \otimes S_{\min}(\Phi_C)$

proved false by Hastings (2009) followed work of Hayden-Winter

later work by King-Fukuda, Brandao-Horodecki

new proofs by Auburn-Szarek-Werner, Collins-Nechita

Shor (2003) showed equiv with other add conj; raised new question

$\rho_A = |\psi_A\rangle\langle\psi_A|, \; \rho_C = |\psi_C\rangle\langle\psi_C|$ local min $S[\Phi_A(\rho_A)], \; S[\Phi_C(\rho_C)]$

$\Rightarrow \quad |\psi_A \otimes \psi_C\rangle\langle\psi_A \otimes \psi_C|$ give local min $S[(\Phi_A \otimes \Phi_C)(\rho_{AC})]$??

$S(\rho) = -\mathrm{Tr}\, \rho \log \rho$ von Neumann entropy $S_{\min}(\Phi) = \min_{\rho} S[\Phi(\rho)]$

quantum channel $\Phi_A : \mathcal{B}(\mathcal{H}_A) \mapsto: \mathcal{B}(\mathcal{H}_B)$ CPT map

completely positive, trace-preserving

Equivalent question via Stinespring

minimum over convex set of density matrices $\rho \geq 0, \quad \text{Tr}\,\rho = 1$

$S[\Phi(\rho)]$ concave $\Rightarrow$ all minima on boundary $\rho = |\psi\rangle\langle\psi|$ pure

Stinespring: $\Phi(\rho) = \text{Tr}_E\, V\rho V^*$ $\qquad V^* V = I_B$

$\Phi(|\psi\rangle\langle\psi|) = V|\psi\rangle\langle\psi|V^*$ $\qquad V : \mathcal{H}_A \mapsto \mathcal{H}_B \otimes \mathcal{H}_E$

Rewrite problem $x \simeq V|\psi\rangle$ vec in subspace of $\mathcal{H}_B \otimes \mathcal{H}_E \simeq \mathcal{H}_A$

rep x as $d_B \times d_E$ matrix

$$\Phi(|\psi\rangle\langle\psi|) = \text{Tr}_E\, V|\psi\rangle\langle\psi|V^* = x\,x^* = \begin{pmatrix} & x & \end{pmatrix} \begin{pmatrix} \\ x^* \\ \\ \end{pmatrix} = \begin{pmatrix} \rho \end{pmatrix}$$

generic situation $d_A^2 \geq d_E >> d_B$ and $x\,x^*$ full rank (non-sing)

loc min $S(xx^*) = -\text{Tr}\, xx^* \log xx^*$ $\quad x \in$ subspace $\mathcal{H}_B \otimes \mathcal{H}_E \simeq \mathcal{H}_A$

Using integral representation to differentiate entropy

$$\log(\rho + t\gamma) - \log \rho = \int_0^\infty \left[\frac{1}{\rho + u} - \frac{1}{\rho + t\gamma + u}\right] du$$

$$= \int_0^\infty \left[\frac{1}{\rho + u}\frac{\rho + t\gamma + u}{\rho + t\gamma + u} - \frac{\rho + u}{\rho + u}\frac{1}{\rho + t\gamma + u}\right] du$$

$$= \int_0^\infty \frac{1}{\rho + u}[\rho + t\gamma + u - \rho - u)]\frac{1}{\rho + t\gamma + u} du$$

$$= t\int_0^\infty \frac{1}{\rho + u}\gamma\frac{1}{\rho + t\gamma + u} du$$

$$= t\int_0^\infty \frac{1}{\rho + u}\gamma\frac{1}{\rho + u} du + t^2\int_0^\infty \frac{1}{\rho + u}\gamma\frac{1}{\rho + u}\gamma\frac{1}{\rho + u} du + O(t^3)$$

First derivative

$$S(\rho + t\gamma) - S(\rho) = -\mathrm{Tr}\,\rho\big[\log(\rho + t\gamma) - \log\rho\big] - t\mathrm{Tr}\,\gamma\log\rho + O(t^2)$$

$$\begin{aligned}
\lim_{t\to 0}\frac{S(\rho + t\gamma) - S(\rho)}{t} &= -\mathrm{Tr}\,\rho\int_0^\infty \frac{1}{\rho + u}\gamma\frac{1}{\rho + u}du - \mathrm{Tr}\,\gamma\log\rho \\
&= -\mathrm{Tr}\,\rho\rho^{-1}\gamma - \mathrm{Tr}\,\gamma\log\rho \\
&= -\mathrm{Tr}\,\gamma - \mathrm{Tr}\,\gamma\log\rho \\
&= 0 - \mathrm{Tr}\,\gamma\log\rho \;=\; -\mathrm{Tr}\,\gamma\log\rho
\end{aligned}$$

since $\mathrm{Tr}\,\gamma = 0$. Valid even when ρ singular.

Later interest: $\mathrm{Tr}\,\int_0^\infty \rho\frac{1}{\rho+u}\gamma\frac{1}{\rho+u}du \;=\; 0$

Expansion in power of t

$$
\begin{aligned}
S(\rho + t\gamma_0 + t^2\gamma_1) &= -\mathrm{Tr}\,(\rho + t\gamma_0 + t^2\gamma_1)\big[\log\rho + tF + t^2 G\big] \\
&= S(\rho) - t\,\mathrm{Tr}\,\gamma_0 \log\rho - t^2 \mathrm{Tr}\,\gamma_1 \log\rho \\
&\quad - t^2 \mathrm{Tr} \int_0^\infty \gamma_0 \frac{1}{\rho + uI} \gamma_0 \frac{1}{\rho + uI} du \\
&\quad + t^2\, \mathrm{Tr} \int_0^\infty \frac{\rho}{\rho + uI} \gamma_0 \frac{1}{\rho + uI} \gamma_0 \frac{1}{\rho + uI} du + O(t^3) \\
&= S(\rho) - t\,\mathrm{Tr}\,\gamma_0 \log\rho - t^2 \mathrm{Tr}\,\gamma_1 \log\rho \\
&\quad - \tfrac{1}{2}\, t^2 \mathrm{Tr} \int_0^\infty \gamma_0 \frac{1}{\rho + uI} \gamma_0 \frac{1}{\rho + uI} du + O(t^3)
\end{aligned}
$$

expect 3 t^2 terms

commute: two ints $-\mathrm{Tr}\,\gamma_0 \rho^{-1} \gamma_0 + \frac{1}{2} \mathrm{Tr}\,\gamma_0 \rho^{-1} \gamma_0 = -\frac{1}{2} \mathrm{Tr}\,\gamma_0 \rho^{-1} \gamma_0$

Surprise: Can still combine in non-comm case

Second derivative

$$\frac{(x+ty)(x+ty)^*}{1+t^2} = xx^* + t(xy^* + yx^*) + t^2(yy^* - xx^*)$$

$$\frac{d^2}{dt^2} S(\rho + t\gamma_0 + t^2\gamma_1)\Big|_{t=0} = \frac{d^2}{dt^2} S\left(\tfrac{(x+ty)(x+ty)^*}{1+t^2}\right)\Big|_{t=0} \equiv D_2[x,y]$$

$$= \quad -\mathrm{Tr}\,\gamma_1 \log\rho - \tfrac{1}{2}\mathrm{Tr}\int_0^\infty \gamma_0 \frac{1}{\rho + uI}\gamma_0 \frac{1}{\rho + uI} du$$

First term $\quad \mathrm{Tr}\,\gamma_1 \log\rho = -\mathrm{Tr}\, yy^* \log xx^* + S(xx^*)$ $\quad$ additive

Concentrate on second — will rewrite beginning with

$$\mathrm{Tr}\int_0^\infty \gamma_0 \frac{1}{\rho + uI}\gamma_0 \frac{1}{\rho + uI} du = \mathrm{Tr}\,\gamma_0 \frac{L_{\log P} - R_{\log P}}{L_P - R_P}\gamma_0$$

Jon Yard independently obtained this form of second derive.

Aside on left, right mult and modular operator

$d \times d$ matrices form Hilbert space with $\langle A, B \rangle = \text{Tr}\, A^* B$

Def. Left and Right mult as linear operators on this vector space

$$L_A(X) = AX \qquad \text{and} \qquad R_B(X) = XB$$

a) L_A and R_B commute $L_A[R_B(X)] = AXB = R_B[L_A(X)]$

b) $A = A^* \;\Rightarrow\; L_A, R_A$ self-adjoint wrt H-S inner prod

For $A, B > 0$ positive definite

c) L_A, R_A pos def $\quad \langle X, R_A(X) \rangle = \text{Tr}\, X^* X A = \text{Tr}\, X A X^* \geq 0$

d) $(L_A)^{-1} = L_{A^{-1}}, \quad (R_B)^{-1} = R_{B^{-1}}$

e) $f(L_A) = L_{f(A)} \qquad \log L_A = L_{\log A} \quad \log R_A = R_{\log A}$

simple form of deep idea: Araki $\Delta_{AB} = L_A R_B^{-1}$ relative modular op

Rewrite using modular operator

$$
\begin{aligned}
\big(L_{\log P} - R_{\log P}\big) A &= A(\log P) - A \log P \\
&= \int_0^\infty \left(A \frac{1}{P + uI} - \frac{1}{P + uI} A \right) du \\
&= \int_0^\infty \frac{1}{P + uI} \big(PA - AP \big) \frac{1}{P + uI} du \\
&= \int_0^\infty \frac{1}{P + uI} (L_P - R_P) A \frac{1}{P + uI} du
\end{aligned}
$$

$$
\frac{L_{\log P} - R_{\log P}}{L_P - R_P} A = \int_0^\infty \frac{1}{P + uI} A \frac{1}{P + uI} du
$$

$P \mapsto \rho \qquad A \mapsto \gamma_0$

$$
\mathrm{Tr} \int_0^\infty \gamma_0 \frac{1}{\rho + uI} \gamma_0 \frac{1}{\rho + uI} du = \mathrm{Tr}\, \gamma_0 \frac{L_{\log P} - R_{\log P}}{L_P - R_P} \gamma_0
$$

Use form of γ_0 to rewrite further

$$\gamma_0 = x\,y^* + y\,x^* \qquad x, y \quad d_B \times d_E, \quad d_E >> d_B \text{ in general}$$

$$\text{assume} \quad \rho = x\,x^* = \begin{pmatrix} & x & \end{pmatrix} \begin{pmatrix} \\ x^* \\ \\ \end{pmatrix} = \begin{pmatrix} \; \end{pmatrix} \quad \text{non-sing}$$

$$x = \sqrt{\rho}\,P = \sqrt{x\,x^*}\,P \qquad PP^* = I_B \qquad x\,P^* = \begin{pmatrix} \sqrt{x\,x^*} & 0 \end{pmatrix}$$

$$\gamma_0 = \sqrt{\rho}\,Py^* + yP^*\sqrt{\rho} \quad \text{Write } Py^* = w + iz \qquad w = w^*, z = z^*$$

$$\gamma_0 = (L_{\sqrt{\rho}} + R_{\sqrt{\rho}})w + i(L_{\sqrt{\rho}} - R_{\sqrt{\rho}})z \qquad \text{After some work get}$$

$$\begin{aligned} \tfrac{1}{2}\mathrm{Tr} \int_0^\infty \gamma_0 \tfrac{1}{\rho + uI} \gamma_0 \tfrac{1}{\rho + uI}\,du &= \tfrac{1}{2}\mathrm{Tr}\,\gamma_0 \tfrac{L_{\log P} - R_{\log P}}{L_P - R_P} \gamma_0 \\ &= \mathrm{Tr}\,w\,\phi(\Delta_{\sqrt{\rho}})(w) + \mathrm{Tr}\,z\,\phi(-\Delta_{\sqrt{\rho}})(z) \end{aligned}$$

$$\phi(a) = \tfrac{1}{2}\tfrac{a+1}{a-1}\log a^2 = \phi(a^{-1}) \qquad \Delta_{\sqrt{\rho}} = L_{\sqrt{\rho}} R_{\sqrt{\rho}}^{-1}$$

Gour-Friedland subadditivity inequality

recall $\phi(a) = \frac{1}{2}\frac{a+1}{a-1}\log a^2 = \phi(a^{-1})$ well-defined for $a \in \mathbf{R},\ a \neq 0$

$$\phi(\pm ac) \ \leq \ \tfrac{1}{2}\big[\phi(a) + \phi(-a) + \phi(c) + \phi(-c)\big]$$

Use spectral theorem for $\Delta = L_{\sqrt{\rho}} R_{\sqrt{\rho}}^{-1}$ Write Δ_A for x_A etc.

$\phi(\pm\Delta_A \otimes \Delta_C)$ as operator inequality

$$\leq \ \tfrac{1}{2}\big[\phi(\Delta_A)\otimes\mathcal{I}_C + \phi(-\Delta_A)\otimes\mathcal{I}_C + \mathcal{I}_A\otimes\phi(\Delta_C) + \mathcal{I}_A\otimes\phi(-\Delta_C)\big]$$

$$\operatorname{Tr} y_{AB}^*\,\phi(\pm\Delta_A\otimes\Delta_C)(y_{AB}) \ \leq \ \tfrac{1}{2}\big[\operatorname{Tr} y_{AB}^*\,\phi(\Delta_A)\otimes\mathcal{I}_C(y_{AB}) + \ldots\big]$$

$$\operatorname{Tr} y_A\otimes y_C\,\phi(\pm\Delta_A\otimes\Delta_C)(y_A\otimes y_C) \ \leq \ \tfrac{1}{2}\Big[\big(\operatorname{Tr}_A y_A\,\phi(\Delta_A) y_A\big)\operatorname{Tr}_C y_C^* y_C + \ldots\Big]$$

Decompose derivative

$$D_2[x,y] \equiv -\operatorname{Tr} yy^* \, \log xx^* - S(xx^*) - \operatorname{Tr} w\, \phi(\Delta_{\sqrt{\rho}})(w) + \operatorname{Tr} z\, \phi(-\Delta_{\sqrt{\rho}})(z)$$

$Py^* \equiv w + iz$ or $\quad P = (U_B \;\, 0), \quad y = (y_{11} \;\, y_{12}) = \begin{pmatrix} w + iz & y_{12} \end{pmatrix}$

$$1 = \operatorname{Tr} yy^* = \operatorname{Tr} y_{11}y_{11}^* + \operatorname{Tr} y_{12}y_{12}^* = \operatorname{Tr} w^2 + \operatorname{Tr} z^2 + \operatorname{Tr} y_{12}^* y_{12}$$

$$S(x^*x) = \operatorname{Tr} w^2 S(x^*x) + \operatorname{Tr} z^2 S(x^*x) + \operatorname{Tr} y_{12}^* y_{12} S(x^*x)$$

$$D_2[x,y] = D_2[x,y_{12}] + D_2[x,w] + D_2[x,iz]$$

convex decomp $\quad D_2[x,y_{12}] = (\operatorname{Tr} y_{12}y_{12}^*) D_2[x,\widetilde{y}_{12}] \qquad \widetilde{y}_{12} = \frac{y_{12}}{\mathrm{Tr} y_{12} y_{12}^*}$

Suff. cond. is that each of above terms > 0 but not nasc

Need $y = \begin{pmatrix} w + iz & y_{12} \end{pmatrix} \in \widehat{\mathcal{H}}_A \subset \mathcal{H}_{B_A} \otimes \mathcal{H}_{E_A}$

but useful to analyze each term separately

Product perturbation

very special case $y_{AC} = y_A \otimes y_C$ not trivial

$$
\begin{aligned}
(P_A \otimes P_C)(y_A \otimes y_C)^* &= (w_A + iz_A) \otimes (w_C + iz_C) \\
&= (w_A \otimes w_C - z_A \otimes z_C) + i(w_A \otimes z_C + z_A \otimes w_C) \\
&= w_{AC} + iz_{AC}
\end{aligned}
$$

Apply inequality, cancel terms, and recombine etc.

neg. contrib to second deriv from $-\frac{1}{2}\mathrm{Tr}\, w_{AC}\phi(\Delta_{AB})w_{AC} \ldots$ etc.

$$
\begin{aligned}
D_2[x_A \otimes x_C, y_A \otimes y_C] \quad \geq \quad & D_2[x_A, e^{i\pi/4}w_A] + D_2[x_A, e^{i\pi/4}z_A] \\
& + D_2[x_C, e^{i\pi/4}w_C] + D_2[x_C, e^{i\pi/4}z_C] \\
\\
= \quad & \tfrac{1}{2}D_2[x_A, w_A + iz_A] + \tfrac{1}{2}D_2[x_A, z_A + iw_A] \\
& + \tfrac{1}{2}D_2[x_C, w_C + iz_C] + \tfrac{1}{2}D_2[x_C, z_C + iw_C]
\end{aligned}
$$

Reduce general case to product

Aside: Need to write using $D_2[x_A, w_A + iz_A]$ etc. so that

$\big(w_A + iz_A, (y_A)_{12}\big) \in$ subspace of allowed perturb.

combine with additive terms to get > 0

$y_{AC} \in (x_A \otimes x_C)^\perp \subset \widehat{\mathcal{H}}_A \otimes \widehat{\mathcal{H}}_C \qquad \mathcal{H}_A = \widehat{\mathcal{H}}_A \subset \mathcal{H}_{B_A} \otimes \mathcal{H}_{E_A}$

Can write $\quad y_{AC} = \sum_j \mu_j\, y_A^j \otimes y_C^j$

Can show cross terms cancel and

$$\begin{aligned} D_2[x_A \otimes x_C, y_{AC}] &= \sum_j \mu_j\, D_2[x_A \otimes x_C, y_A^j \otimes y_C^j] \\ &= \sum_j \mu_j\, (>0) \;>\; 0 \end{aligned}$$

Subtle point

development assumed $\operatorname{Tr} x\, y^* = 0$ and $\operatorname{Tr} xx^* = \operatorname{Tr} yy^* = 1$.

But $\quad \operatorname{Tr} (x_A \otimes x_C)(y_A \otimes y_C)^* = (\operatorname{Tr}_A x_A y_A^*)(\operatorname{Tr}_C x_C y_C^*) = 0$

Could have $\operatorname{Tr}_A x_A y_A^* \neq 0$, in which case

$$y_A = s\, x_A + \sqrt{1-s^2}\, \widetilde{y}_A \qquad \operatorname{Tr}_A x_A\, \widetilde{y}_A^* = 0, \quad \operatorname{Tr}_A \widetilde{y}_A \widetilde{y}_A \widetilde{y} = 1$$

$$\frac{x_A + t y_A}{\sqrt{1+t^2}} \mapsto \frac{(1+s)\, x_A + \sqrt{1-s^2}\, t\, \widetilde{y}_A}{\sqrt{(1+s)^2 + (1-s^2)t^2}} = \frac{x_A + \tau \widetilde{y}_A}{\sqrt{1+\tau^2}}$$

$t \mapsto \quad \tau = \sqrt{\frac{1-s}{1+s}}\, t \quad$ only effect is to rescale t

Doesn't affect sign of derivative $\qquad$ replace μ_j by $\sqrt{\frac{1-s_j}{1+s_j}}\mu_j$

Modifications when xx^* singular

Block $x \to \begin{pmatrix} x_{11} & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \to \begin{pmatrix} x_{11} & 0 \\ 0 & 0 \end{pmatrix}$ range xx^* / ker xx^*

combined cols for $\ker xx^*$ and $\mathcal{H}_E \backslash \mathcal{H}_B$ $\quad y = \begin{pmatrix} y_{11} & y_{12} \\ y_{21} & y_{22} \end{pmatrix}$

first row $\begin{pmatrix} y_{11} & y_{12} \end{pmatrix}$ same as non-sing. $\quad P^* y = y_{11}$

$y_{22} \neq 0$ formally $-\text{Tr}\, y_{22} y_{22}^* \log xx^* \to +\infty$

Majorization arg shows $y_{22} \neq 0$ always increases entropy

y_{21} big problem: $\quad x \to x + \epsilon Q \qquad QQ^* = \text{proj}(\ker xx^*)^\perp$

go back to $-\text{Tr}\, y^* y \log x^* x - \frac{1}{2} \text{Tr} \int_0^\infty \gamma_0 \frac{1}{\rho + uI} \gamma_0 \frac{1}{\rho + uI} du$

get $-\text{Tr}\, y_{21} y_{21}^* \log \epsilon + \text{Tr}\, y_{21} y_{21}^* \log \epsilon - \text{Tr}\, y_{21}^* y_{21} \log(xx^* + \epsilon QQ^*)$

$\to -\text{Tr}\, y_{21}^* y_{21} \log xx^*$ well-behaved and additive

Conjecture for real subspace of self-adjoint matrices

GF big deal of Gurvits "real" counter-ex and role of **C** vs **R**.

Gurvits: subspace is span $\{I, i\sigma_y\}$ anti-sym $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$.

some kind of completely noisy channel not qubit restrict to **R**

But anti-sym not preserved over $\otimes$ get inputs with output $\neq \frac{1}{4}I$.

Special case $d_B = d_E$ (very atypical) more natural to consider for **R**

subspace of real symmetric (or even self-adjoint) matrices.

Conj: Local additivity holds over **R** in this case

Need slightly different inequality which seems OK numerically