Entropy of Stabilizer States

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Classical Shannon Entropy

Def: $H(p) = -\sum_k p_k \log p_k$ Shannon (1948)

Properties of multi-party systems:

Pos $H(p) \geq 0$

SSA $H(A) + H(B) - H(A \cap B) - H(A \cup B) \geq 0$

Mono $A \subset B \quad \Rightarrow \quad H(A) \leq H(B)$

$H(A) \equiv H(p_A)$ etc.

$A, B, \ldots$ subsets of some index set $\mathcal{X} \simeq [1, 2, , \ldots N]$

Quantum Entropy

Def: (1927) von Neuman Entropy of quantum state ρ

density matrix $\rho \geq 0, \;\text{Tr}\,\rho = 1$

$$S(\rho) = -\text{Tr}\,\rho\log\rho = -\sum_k \lambda_k \log\lambda_k$$

Props: 1) $S(\rho) \geq 0$ 2) $S(\rho)$ concave

3) SSA for multi-party systems $\mathcal{H} = \mathcal{H}_A \otimes \mathcal{H}_B \otimes \mathcal{H}_C$

$$S(\rho_{ABC}) + S(\rho_B) \leq S(\rho_{AC}) + S(\rho_{BC})$$

Quant marginals or reduced density matrix $\rho_A = \text{Tr}_B\,\rho_{AB}$

That's all folks!

Conditional information

Cond Info $\quad S(\rho_{AB}) - S(\rho_A)$ concave in ρ_{AB}

Cond Info always ≥ 0 for classical systems

Can have quant um state $|\psi_{AB}\rangle = \frac{1}{\sqrt{2}}\big(|00\rangle + |11\rangle\big) \in \mathcal{H}_A \otimes \mathcal{H}_B$

$\rho_A = \text{Tr}_B\, \rho_{AB} = \text{Tr}_B\, |\psi_{AB}\rangle\langle\psi_{AB}| = \frac{1}{2} I_A$ max mixed

Cond Info $= 0 - \log 2 < 0$ $\qquad \rho_{AB} = |\psi_{AB}\rangle\langle\psi_{AB}|$ pure

Cond Info neg for highly entangled quantum states

once thought "defect"; now has nice info theory interp.

conditional info is amount of info need to learn AB knowing A

when neg, measures entanglement available for future info trans

M. Horodecki, Oppenheim and Winter (2005) state merging

Weak Monotonicity

$\rho_{AD} = |\psi_{AD}\rangle\langle\psi_{AD}|$ pure $\Rightarrow$ $S(\rho_A) = S(\rho_D)$

pure state is rank one projection op, $\rho_{AD}^2 = \rho_{AD} \geq 0$

Purification: Given ρ_{ABC} can find vector $|\psi_{ABCD}\rangle$ s.t

$\rho_{ABC} = \mathrm{Tr}_D\, |\psi_{ABCD}\rangle\langle\psi_{ABCD}|$

Apply to SSA $S(\rho_{AB}) + S(\rho_{BC}) - S(\rho_{ABC}) - S(\rho_B) \geq 0$

Equiv. ineq: $S(\rho_{CD}) + S(\rho_{BC}) - S(\rho_D) - S(\rho_B) \geq 0$

Weak monotonicity or "monogamy of entanglement"

Cond Info $S(\rho_{BC}) - S(\rho_B)$ and $S(\rho_{CD}) - S(\rho_D)$ can't both be neg

Charlie can be entangled with Beverly or Dorothy, but not both

Purification and Complementarity

Spectral decomp of $\rho_A = \sum_k \lambda_k |\phi_k\rangle\langle\phi_k|$

Let $\{\theta_k\}$ any O.N. basis for $\mathcal{H}_B \simeq \mathcal{H}_A$

Def $|\psi_{AB}\rangle = \sum_k \sqrt{\lambda_k}\, |\phi_k\rangle \otimes |\theta_k\rangle$ "purification"

$\rho_B = \text{Tr}_A\, |\psi_{AB}\rangle\langle\psi_{AB}| = \sum_k \lambda_k |\theta_k\rangle\langle\theta_k|$ same spectrum as ρ_A

vector $|\psi\rangle \in \mathcal{H}$ and rank one proj $|\psi\rangle\langle\psi|$ both called "pure" state

identify class prob vector p_k with diag D.M. $\rho = \sum_k p_k |e_k\rangle\langle e_k|$

can also "purify" class prob dist $|\psi_{AB}\rangle = \sum_k p_k |e_k \otimes f_k\rangle$ quant state

Start with arbitrary $\psi_{AB} = \sum_{jk} a_{jk} |e_j \otimes f_k\rangle$

Use Sing Val Decomp (aka "Schmidt") $\psi_{AB} = \sum_k \mu_k |\phi_k\rangle \otimes |\theta_k\rangle$

non-zero evals of both ρ_A and ρ_B are $\mu_k^2 \;\Rightarrow\; S(\rho_A) = S(\rho_B)$

Essentially AA^* and A^*A same non-zero e-vals

Properties of quantum entropy

Some sense: Only one inequality, SSA

$S(\rho) \geq 0$ is really just normalization condition

most purposes only need consistency, $\mathrm{Tr}_{AB}\, \rho_{AB} = \mathrm{Tr}_{\,A}\, \rho_A$

But we need it to so that entropy vectors form cone

Have seen Weak Monotonicity is equiv, to SSA in quantum setting

Even concavity not indep: clever choice of block matrix

sub add $S(\rho_{AB}) \leq S(\rho_B) + S(\rho_B) \quad \Rightarrow$ concavity

similarly SSA $\Rightarrow$ Cond Info concave in ρ_{AB}

But these are not linear implications, so will need to add something

N-party Entropy Cones

$\rho_{12\ldots N}$ N-party state

consider all reduced states $\rho_1, \rho_2, \ldots, \rho_{12}, \ldots \rho_{37} \ldots \rho_{234} \ldots$

fix order and generate vector in $\mathbf{R}^{2^N}$ from entropies

$$\Big(S(\rho_1), S(\rho_2), \ldots S(\rho_{12}), \ldots S(\rho_{37}), \ldots S(\rho_{234}), \ldots, \Big)$$

closure of all such vectors is a convex cone – entropy cone

classical entropy cone $\subsetneq$ quantum entropy cone

would like to characterize these cones, esp. quantum cones

Cone in $\mathbf{R}^{2^N}$ generated by half-planes from various inequalities

Shannon cone: Pos, SSA, Mono

YZ: Shannon Ent Cone $\supsetneq$ Classical Ent Cone for $N > 3$

cones of entropy type vectors

$A, B, \ldots$ subsets of some index set $\mathcal{X} \simeq [1, 2, , \ldots N]$

$J = \{A, C, D, \ldots\}$ set of substes $J^C = \{B \in \mathcal{X} : B \notin J\}$

$\overline{\Sigma}_N^C$ and $\overline{\Sigma}_N^Q$ closure of cone of N-party entropy vectors

Γ_N^C polymatroid $H(p) \geq 0$, $H(AB) > H(A)$, SSA

Γ_N^Q polyquantoid $S(\rho) \geq 0$, weak mono, SSA

or $S(\rho) \geq 0$, SSA , and

quant marginals of $(N+1)$-party states $S(\rho_J) = S(\rho_{J^C})$

Λ_N^C and Λ_N^Q add linear rank ineq to Γ_N^C and Γ_N^Q

Can completely characterize $\Lambda_4^Q \equiv \Gamma_4^Q$ and Ingelton Ineq.

Don't know if $\overline{\Sigma}_4^Q$ satisfies non-Shannon inequalities

Defs

mutual information

$$I(A:B) \equiv S(A) + S(B) - S(AB)$$

conditional mutual information

$$I(A:B|C) \equiv S(AC) + S(BC) - S(C) - S(ABC)$$

Ingleton expression

$$\mathrm{ING}(AB:CD) \equiv I(A:B|C) + I(A:B|D) + I(C:D) - I(A:B)$$

SSA equiv to $I(A:B|C) \geq 0$

Ingleton inequality $\mathrm{ING}(AB:CD) \geq 0$

not universal – simplest "linear rank inequality"

Examples of "balanced" inequality – number of $A, B, \ldots$ cancel out

Group rank inequalities

Thm: (Chan-Yeung) There is a 1-1 correspondence between entropy inequalities for classical N-party systems and inequalities for the sizes of subgroups of groups.

Ex: SSA equiv to $|G_1| \cdot |G_2| \leq |G_1 \cap G_2| \cdot |G|$

Pf Idea: Can find class prob dist with entropy of marginals $\log \frac{|G|}{|G_J|}$

Subgroups with special properties, e.g., normal or abelian,
may satisfy additional inequalities

linear rank inequalities – sizes of subspaces of vector spaces

G_A and G_B normal $\Rightarrow \mathrm{ING}(AB : CD) \geq 0.$

Ingleton is only linear rank inequality for 4-party systems

non-Shannon inequalities

Classical N-party entropy cone satisfies non-Shannon ineq.

- Yeung-Zhang (1997-98) gave first $\quad t = 1$
- Dougherty-Freiling, Zeger (2006+)

  found new inequalities by computer search
- Matúš (2007) found two infinite families $t \geq 0$ integer

$$t\,\mathrm{ING}(AB : CD) + I(A : B|D) + \tfrac{t(t+1)}{2}\big[I(B : D|C) + I(C : D|B)\big] \;\geq\; 0$$

$\mathrm{ING}(AB : CD) +$ positive terms $\;\geq 0$

$\Rightarrow$ 4-party entropy cone not polyhedral

suggests don't yet know all classical 4-party inequalities

Know: Classical entropy cone described by Mono and balanced ineq

Ingleton in Quantum Setting

Any of following conditions implies Ingelton inequality

a) $\rho_{ABCD} = |\psi_{ABCD}\rangle\langle\psi_{ABCD}|$ is any pure 4-party state.

b) $\rho_{ABCD} = \rho_{ABC} \otimes \rho_D$ or $\rho_A \otimes \rho_{BCD}$

c) ρ_{ABCD} symmetric under partial exchange between (A, B) and (C, D), under any *one* (but not two) of the exchanges $A \leftrightarrow C$, $B \leftrightarrow D$, $A \leftrightarrow D$ or $B \leftrightarrow C$.

Ingleton Inequality not universal, but hard to find violations

N-party linear rank inequalities

Kinser (2011) found first infinite family

DFZ (2010) found tree algorithm for generating all families

when pair of subsystems with "common information"

have form $\sum + c_k$(cond mutual info) $\geq I(A : B)$

In group set up, pair of normal subgroups $\simeq$ "common info"

Will show $\Rightarrow$ all stabilizer states satisfy such ineq.

BUT Chan, Grant, Kern (2011) showed $\exists$ linear rank ineq.

that are not multi-party Ingleton

suggests DFZ does not give all linear rank ineq.

don't know if stabilizer states would satisfy such ineq.

Common information

State ρ_{AB} of two subsystems A, B has common information if

Can add another party ζ such that

$$H(A\zeta) = H(A), \quad H(B\zeta) = H(B) \text{ and } \qquad H(\zeta) = I(A:B)$$

corresponds to pair of normal subgroups in groups setting

BUT Chan, Grant, Kern (2010) showed $\exists$ other linear rank ineq

Main Result

Thm: Ingleton cone (Pos, SSA, WM, ING) for 4-party systems is precisely the closure of the convex hull of entropy vectors that arise from reduced states of 5-party pure stabilizer states.

Thm: Reduced states of $(N+1)$-party stabilizer state satisfy every N-party linear rank inequality from common information (DFZ).

Thm: (Indep by Gross and Walter) Every balanced classical entropy inequality satisfied by reduced states of stabilizer states.

Weyl-Heisenberg group

Generalized shift and phase operators on $\mathbf{C}_d$

$$X|e_k\rangle = |e_{k+1}\rangle \qquad Z|e_k\rangle = \omega|e_k\rangle \quad \omega = e^{2\pi i/d}$$

$XZ = \omega ZX$ $\quad W$ group gen by $X^j Z^k$

Center $C = \{\omega^k \mathbb{1}\}_{k=0,1,\ldots d-1}$ multiples of identity

$\widehat{W} = W/C$ Abelian – rough prod $X^j Z^k$ ignore phase

Consider unitary group on $\bigotimes_{x \in \mathcal{X}} \mathcal{H}_x$ of form $W = \bigotimes_{x \in \mathcal{X}} W_x$

Stabilizer G Abelian subgroup of W

simultaneous eigenspace is Quantum Error Correction Code

Stabilizer state is simul eigenstate of max Abel subgroup G

Stabilizer states

W_j subgroup of $\mathcal{U}(\mathcal{H}_j)$ with center C_j mult of I (e.g. Weyl-Heis)

$\widehat{W_j} = W_j / C_j$ Abelian with size d_j^2 $\quad d_j = \dim \mathcal{H}_j$.

Consider G max Abelian subgroup of $W = \bigotimes_j W_j$

Simultaneous e-vec of all $g \in G$ called a stabilizer state

Why are one-dim codes interesting? aka graph states,

Important role in one-way quantum computing cluster state

Arise in mutually unbiased bases

$$G_J = \{g = g_j \, g_k : g_k = I, \;\; k \in J^c\} \quad \text{think of } g = g_j \otimes I$$

Key Thm.

Thm: (indep several group ≈ 2004) $\rho = |\psi\rangle\langle\psi|$ pure stab state

$$\rho_J = \mathrm{Tr}_{J^c}|\psi\rangle\langle\psi| \text{ proj of rank } \frac{|\widehat{G_J}|}{d_J} \quad \Rightarrow \quad S(\rho_J) = \log \frac{d_J}{|\widehat{G_J}|}$$

Cor: Since $|\widehat{G}| = d = d_J d_{J^c}$ last eq. can be rewritten as

$$S(\rho_J) = S(\rho_{J^c}) = \log \frac{|\widehat{G}|}{|\widehat{G_{J^c}}|} - \log d_J$$

$\log \frac{|\widehat{G}|}{|\widehat{G_{J^c}}|}$ is a group entropy and

Additional terms cancel for any balanced inequality

Moreover stab group $\widehat{G}$ Abelian $\quad \Rightarrow \quad$ Ingleton holds

$\Rightarrow \quad$ Matúš ineq. = Ingleton + pos terms hold

Classical Balanced Inequalities

More parties – common info assoc with normal subgroups

$\Rightarrow$ all DFZ type linear rnak inequalities hold

More $\Rightarrow$ all balanced classical entropy ineq hold.

D. Gross and M. Walter (arxiv:1302.6902)

independently by different methods

Use phase space methods to find classical prob dist X

s.t. stabilizer states satisfy $S(\rho_J) = H(X_J) - |J|$

$\Rightarrow$ all balanced classical ineq. hold

Sketch Proof: part I

$P = |\psi\rangle\langle\psi|$ proj on simul e-state of G max Abel subgroup

$gP = \chi(g)|\psi\rangle\langle\psi| = \chi(g)P$ $\chi(g)$ character of 1-dim rep.

$$P = |\psi\rangle\langle\psi| = \frac{1}{|G|}\sum_{g\in G}\overline{\chi(g)}\,g = \frac{1}{|G_0|}\sum_{g\in G_0} g$$

$G_0 \simeq G/C$ identify subgp $G_0 \subset G$ with quotient group

$$P^2 = \frac{1}{|G_0|^2}\sum_g\sum_h gh = \frac{1}{|G_0|}\sum_g g = P$$

Aside: trivial rep not essential $P_j = |\psi_j\rangle\langle\psi_j| \equiv \frac{1}{|G_0|}\sum_g \overline{\chi}_j(g)\,g$

$\text{Tr}\, P_j P_k - |\langle\psi_j, \psi_k\rangle|^2 = \delta_{jk}$ O.N. basis of e-states

Sketch Proof: part II:

Suffices to consider bipartite setting

$$\rho_{AB} = |\psi_{AB}\rangle\langle\psi_{AB}| = \frac{1}{|\widehat{G}|} \sum_{g_a \otimes g_B \in G} g_A g_B$$

$\operatorname{Tr} g_B = 0$ unless $g_B = \mathbb{1}$

$$\rho_A = \frac{1}{d_A d_B} \sum_{g_A \otimes \mathbb{1}_B} g_A \, d_B = \frac{|\widehat{G_A}|}{d_A} \left(\frac{1}{|\widehat{G_A}|} \sum_{g_A \in \widehat{G_A}} g_A \right)$$

$$\Rightarrow \rho_A \;\; \text{proj of rank } \tfrac{|\widehat{G_A}|}{d_A} \;\; \Rightarrow \;\; S(\rho_A) = \log \frac{d_A}{|\widehat{G_A}|}$$

subtle point $|\widehat{G}| = d = d_A d_B$ but $|\widehat{G_A}| \neq d_A$

4-party "Ingleton" cone – other direction

Can explicitly compute extreme rays of 4-party Ingleton cone

Show each ray can be realized using a 5-party pure stabilizer state

All but one in (2006) thesis of Ben Ibinson

DFZ methods give all 5-party linear rank inequalities

Conjecture also achieved with 6-party pure stabilizer states

Conj: All DFZ inequalities achieved with pure stabilizer states

How to violate Ingleton

0	0	1	0	0	0
0	1	0	0	0	1
1	0	0	0	1	0
1	1	0	1	1	1

$$\tfrac{1}{4}|1000\rangle\langle 1000| + \tfrac{1}{4}|0111\rangle\langle 0111| + \tfrac{1}{4}|0010\rangle\langle 0010| + \tfrac{1}{4}|0001\rangle\langle 0001|$$

$$\mathrm{ING}(AB:CD) = 0 + 0 + 0 - I(A:B) \leq 0$$

"quantumize" $\quad |\psi\rangle = \frac{1}{\sqrt{2}}\big(|1000\rangle + |0111\rangle\big)$

$$\rho_{ABCD} = \tfrac{1}{2}|\psi\rangle\langle\psi| + \tfrac{1}{4}|0010\rangle\langle 0010| + \tfrac{1}{4}|0001\rangle\langle 0001|$$

same reduced states as classical

Challenge: Find truly quantum state that violates Ingleton

Open questions

- All entropy vectors which violate Ingleton in classical cone??
- Do new classical entropy ineq extend to quantum systems?
- What inequalities characterize quantum entropy cone?
- Do stabilizer states satisfy linear rank inequalities that do not arise from common info ?
- Find an explicit example of such an inequality.
- Do all classical inequalities have form

  linear rank ineq $+$ pos terms ≥ 0?
- How much of a restriction are new inequalities, i.e., relative size of true entropy cone and Shannon or vonNeuman cone