A Unified Treatment of Convexity of Relative Entropy and Related Trace Functions, with Conditions for Equality

simple proofs of SSA and Lieb's concavity of $\mathrm{Tr}\, K^* A^p K B^{1-p}$

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Outline of Talk

- Background
- Proof of convexity
  - Introduce left L_A and right R_B mult ops
  - Integral reps
  - Operator Schwarz inequality with L_A and R_B
  - extend to $q \neq 1 - p$
- Prove corollaries MPT and SSA
- Equality conds
- Revisit Carlen- Lieb inequalities
  - Equality conditions
- Final comments on SSA and gen

Basics

Properties of Relative Entropy: $H(A, B) = \operatorname{Tr} A(\log A - \log B)$

Joint Convexity $H\Big(\sum_j A_j, \sum_j B_j \Big) \leq \sum_j H(A_j, B_j)$

Cor: Monotone under Partial Trace (MPT) $\mathcal{H} = \mathcal{H}_1 \otimes \mathcal{H}_2$

$H(A_1, B_2) \leq H(A_{12}, B_{12})$ ILW: that's all folks!

Cor: Monotone under CPT Map $H[\Phi(A), \Phi(B)] \leq H(A, B)$

(quantum channel)

Special Case of MPT $A_{12} \to A_{123}$, $B_{12} \to I_1 \otimes A_{23}$

$$H(A_{12}, A_2) \leq H(A_{123}, A_{23})$$

$-S(A_{12}) + S(A_2) \leq -S(A_{123}) + S(A_{23})$ $S(A) = -\operatorname{Tr} A \log A$

strong subadditivity (SSA) of quantum entropy

Aside:

G Homogenous of degree one

$G(\lambda A) = \lambda G(A) \quad \Rightarrow \quad G(A)$ convex $\Leftrightarrow$ subadditive

convex $\Rightarrow$ subadditive

$$\frac{1}{2} G(A) = G(\tfrac{1}{2} A + \tfrac{1}{2} B) \leq \frac{1}{2} G(A) + \frac{1}{2} G(B)$$

subadditive $\Rightarrow$ convex

$$\begin{aligned} G\left[xA + (1-x)\right] &\leq G(xA) + G[(1-x)B] \\ &= xG(A) + (1-x)G(B) \end{aligned}$$

Background

Original proof of SSA based on Lieb's result below

WYD skew entropy $\frac{1}{2}\mathrm{Tr}\,[K,\gamma^p][K,\gamma^{1-p}]$

for $K = K^*$ and γ a density matrix

Wigner-Yanase introduced for $p = \frac{1}{2}$ and proved concave in γ.

Dyson suggested $p \in (0,1)$ – led to conjecture

Conj: $\gamma \mapsto \mathrm{Tr}\,K\gamma^p K\gamma^{1-p} - \mathrm{Tr}\,K\gamma K$ concave

Lieb dropped linear term and proved generalization

$(A,B) \mapsto \mathrm{Tr}\,K^* A^p K B^{1-p}$ concave for $p \in (0,1)$

Claim: But advantage to retaining linear term !!

Retaining the linear term

$$J_p(K, A, B) = \frac{1}{p(1-p)} \left[\operatorname{Tr} K^* A K - \operatorname{Tr} K^* A^p K B^{1-p} \right]$$

Note: well-def for $p > 0$ and factor $(1-p)$ changes sign at $p = 1$

Thm: $(A, B) \mapsto J_p(K, A, B)$ is convex for $p \in (0, 2)$

$\Rightarrow \quad \operatorname{Tr} K^* A^p K B^{1-p}$ concave for $p \in (0, 1)$

$\Rightarrow \quad \operatorname{Tr} A(\log A - \log B)$ convex $p = 1$ – extend by cont $K = I$

$\Rightarrow \quad \operatorname{Tr} K^* A^p K B^{1-p}$ convex for $p \in (1, 2]$ and $p \in [-1, 0)$

will give proof, which is elementary, short, and sweet

$\operatorname{Tr} A = \operatorname{Tr} B \quad \Rightarrow \quad J_p(K, A, B) \geq 0$ with equality $\Leftrightarrow A = B$

pseudo-metric in same sense as relative entropy $H(A, B)$ gen Klein

Pedestrian modular operator

$d \times d$ matrices form Hilbert space with $\langle A, B \rangle = \mathrm{Tr}\, A^* B$

Def. Left and Right mult as linear operators on this vector space

$$L_A(X) = AX \qquad \text{and} \qquad R_B(X) = XB$$

a) L_A and R_B commute $L_A[R_B(X)] = AXB = R_B[L_A(X)]$

b) $A = A^* \;\Rightarrow\; L_A, R_A$ self-adjoint wrt H-S inner prod

For $A, B > 0$ positive definite

c) L_A, R_A pos def $\langle X, R_A(X) \rangle = \mathrm{Tr}\, X^* X A = \mathrm{Tr}\, X A X^* \geq 0$

d) $(L_A)^{-1} = L_{A^{-1}}, \quad (R_B)^{-1} = R_{B^{-1}}$

e) $f(L_A) = L_{f(A)} \quad f(R_B) = R_{f(B)}$, e.g., $L_A^p = L_{A^p}, R_A^p = R_{A^p}$

simple form of deep idea: Araki $\Delta_{AB} = L_A R_B^{-1}$ relative modular op

Aside: Functions of operators

For $A = UDU^*$, define $f(A) = U\,f(D)\,U^*$

$$A = U \begin{pmatrix} \lambda_1 & 0 & \ldots & 0 \\ 0 & \lambda_2 & \ldots & 0 \\ \vdots & & \ddots & \vdots \\ 0 & \ldots & 0 & \lambda_d \end{pmatrix} U^* \quad f(A) = U \begin{pmatrix} f(\lambda_1) & 0 & \ldots & 0 \\ 0 & f(\lambda_2) & \ldots & 0 \\ \vdots & & \ddots & \vdots \\ 0 & \ldots & 0 & f(\lambda_d) \end{pmatrix} U^*$$

equiv. to any reasonable def using power series, integral rep., etc.

also applies to operators, e.g., L_A acting on M_d space of matrices

$$A|\phi_j\rangle = \alpha_j|\phi_j\rangle \quad \Rightarrow \quad L_A|\phi_j\rangle\langle\phi_k| = \alpha_j|\phi_j\rangle\langle\phi_k|$$

$k = 1, 2, \ldots d$

e-vals of L_A deg

$$f(A)|\phi_j\rangle = \alpha_j|\phi_j\rangle \qquad f(L_A)|\phi_j\rangle\langle\phi_k| = f(\alpha_j)|\phi_j\rangle\langle\phi_k|$$

In particular, $\mathrm{Tr}\, L_{\log Q}(P) = \mathrm{Tr}\,(\log L_Q)(P)$

Back to $J_p(K, A, B)$

$$g_p(x) = \begin{cases} \frac{1}{p(1-p)}(x - x^p) & p \neq 1 \\ x \log x & p = 1 \end{cases}.$$

well-defined for $x > 0$ and $p \neq 0$, but $p \in [\frac{1}{2}, 2]$ would suffice

$$J_p(K, A, B) \;\equiv\; \operatorname{Tr} \sqrt{B} K^* \, g_p\big(L_A R_B^{-1}\big)(K\sqrt{B})$$

$$= \begin{cases} \frac{1}{p(1-p)}\big(\operatorname{Tr} K^* A K - \operatorname{Tr} K^* A^p K B^{1-p}\big) & p \in (0,1) \cup (1,2) \\ \operatorname{Tr} K K^* A \log A - \operatorname{Tr} K^* A K \log B) & p = 1 \\ -\frac{1}{2}\big(\operatorname{Tr} K^* A K - \operatorname{Tr} A K B^{-1} K^* A\big) & p = 2 \end{cases}$$

$$J_1(I, A, B) = \operatorname{Tr} A(\log A - \log B) = H(A, B)$$

Aside: extend to $[-1, 0)$

not quite symmetric around $p = \frac{1}{2}$ $\quad p \leftrightarrow 1-p$

$$\widetilde{g}_p(x) = w g_{1-p}(w^{-1}) = \begin{cases} \frac{1}{p(1-p)}(1 - x^p) & p \neq 0 \\ -\log x & p = 0 \end{cases} \qquad p \in [-1, 1)$$

$$J_p(K, B, A) = \widetilde{J}_{1-p}(K^*, A, B)$$

$\widetilde{J}_p(K, A, B)$ jointly convex for $p \in [-1, 1)$

$$\widetilde{J}_0(I, A, B) = \operatorname{Tr} B(\log B - \log A) = H(B, A)$$

Integral representations

$$\frac{g_p(x)}{x} = \begin{cases} \frac{1}{p(1-p)}(1 - x^{p-1}) & p \neq 1 \\ \log x & p = 1 \end{cases}$$

well-def for $x \in (0, \infty)$ and operator monotone for $p \in (0, 2]$, or,

anal cont to upper half of complex plane and UHP $\mapsto$ UHP

$\Rightarrow g_p(x)$ has integral rep of form

$$\begin{aligned} g_p(x) &= ax + \int_0^\infty \frac{x^2 t - x}{x + t}\, \nu(t)\, dt \\ &= ax + \int_0^\infty \left[\frac{x^2}{x+t} - \frac{1}{t} + \frac{1}{x+t}\right] t\nu(t)\, dt \end{aligned}$$

with $\nu(t) \geq 0$

Specific integrals – elementary

$$\int_0^\infty \frac{x^{p-1}}{x+1} = \frac{\pi}{\sin p\pi} \qquad 0 < p < 1 \qquad\qquad c_p = \frac{\sin p\pi}{\pi}$$

allows us to give the following explicit representations

$$g(x) = \begin{cases} \frac{1}{p(1-p)}\left[x + c_p \int_0^\infty \left(\frac{t}{x+t} - 1\right) t^{p-1} dt\right] & p \in (0,1) \\ \int_0^\infty \left(\frac{x^2}{x+t} - 1 + \frac{t}{x+t}\right) \frac{1}{1+t} dt & p = 1 \\ \frac{1}{p(1-p)}\left[x + c_{p-1} \int_0^\infty \frac{x^2}{x+t} t^{p-2} dt\right] & p \in (1,2) \\ \frac{1}{2}(-x + x^2) & p = 2 \end{cases}$$

Important: For $p \in (0,2)$ integrand supported on $(0,\infty)$.

Integal representation using L_A and R_B

Recall $J_p(K,A,B) \;=\; \mathrm{Tr}\,\sqrt{B}K^*\, g_p\big(L_A R_B^{-1}\big)(K\sqrt{B})$

$$g_p(x) = ax + \int_0^\infty \left[\tfrac{x^2}{x+t} - \tfrac{1}{t} + \tfrac{1}{x+t}\right] t\nu(t)\,dt$$

$$\begin{aligned}\mathrm{Tr}\,\sqrt{B}K^*\frac{1}{L_A R_B^{-1} + tI}(K\sqrt{B}) &= \mathrm{Tr}\,\sqrt{B}K^*\frac{R_B}{L_A + tR_B}(K\sqrt{B}) \\ &= \mathrm{Tr}\,BK^*\frac{1}{L_A + tR_B}(KB)\end{aligned}$$

$$\begin{aligned}J_p(K,A,B) \;=\; &\mathrm{Tr}\,K^*AK - \mathrm{Tr}\,KBK^*\int_0^\infty \nu(t)\,dt \\ &+ \int_0^\infty \left[\mathrm{Tr}\,K^*A\frac{1}{L_A + tR_B}(AK) + \mathrm{Tr}\,BK^*\frac{1}{L_A + tR_B}(KB)\right] t\,\nu(t)\,dt\end{aligned}$$

Suffices to show $(A,B,X) \mapsto \mathrm{Tr}\,X^*\dfrac{1}{L_B + tR_A}(X)$ jointly convex

Proof:

Note: $\operatorname{Tr}(\lambda X)^* \dfrac{1}{L_{\lambda B} + tR_{\lambda A}}(\lambda X) = \lambda \operatorname{Tr} X^* \dfrac{1}{L_B + tR_A}(X)$

Homo of degree 1 $\Rightarrow$ suffices to prove subadditivity

Let: $\quad M = (\ )^{-1/2}(X) - (\ )^{1/2}(\Lambda)$

$$\begin{aligned} \operatorname{Tr} M^* M &= \langle M, M \rangle \\ &= \langle [(\ )^{-1/2}(X) - (\ )^{1/2}(\Lambda)]\,,\ [(\ )^{-1/2}(X) - (\ )^{1/2}(\Lambda)] \rangle \\ &= \langle X, (\ )^{-1}(X) \rangle - \langle X, \Lambda \rangle - \langle \Lambda, X \rangle + \langle \Lambda, (\ )(\Lambda) \rangle \end{aligned}$$

Choose $M = (L_A + tR_B)^{-1/2}(X) - (L_A + tR_B)^{1/2}(\Lambda)$

$\operatorname{Tr} M^* M =$

$$\operatorname{Tr} X^*(L_A + tR_B)^{-1}(X) - \operatorname{Tr} X^*\Lambda - \operatorname{Tr} \Lambda^* X + \operatorname{Tr} \Lambda^*(L_A + tR_B)(\Lambda)$$

Let $M_j = (L_{A_j} + tR_{B_j})^{-1/2}(X_j) - (L_{A_j} + tR_{B_j})^{1/2}(\Lambda)$. Then

$$0 \leq \sum_j \operatorname{Tr} M_j^* M_j = \sum_j \operatorname{Tr} X_j^* (L_{A_j} + tR_{B_j})^{-1}(X_j)$$
$$-\operatorname{Tr}\left(\sum_j X_j^*\right)\Lambda - \operatorname{Tr}\Lambda^*\left(\sum_j X_j\right) + \operatorname{Tr}\Lambda^* \sum_j \left(L_{A_j} + tR_{B_j}\right)\Lambda$$

Choose $\Lambda = \dfrac{1}{L_{\sum_j A_j} + tR_{\sum_j B_j}}\left(\sum_j X_j\right)$. Use $\sum_j L_{A_j} = L_{\sum_j A_j}$

$$\begin{aligned} \operatorname{Tr}\Lambda^* \sum_j \left(L_{A_j} + tR_{B_j}\right)\Lambda &= \operatorname{Tr}\left(\sum_j X_j^*\right) \tfrac{1}{L_{\sum_j A_j} + tR_{\sum_j B_j}} \left(\sum_j X_j\right) \\ &= \operatorname{Tr}\left(\sum_j X_j^*\right)\Lambda = \operatorname{Tr}\Lambda^*\left(\sum_j X_j\right) \end{aligned}$$

$$0 \leq \sum_j \operatorname{Tr} X_j^* \frac{1}{L_{A_j} + tR_{B_j}}(X_j) - \operatorname{Tr}\left(\sum_j X_j^*\right) \tfrac{1}{L_{\sum_j A_j} + tR_{\sum_j B_j}} \left(\sum_j X_j\right)$$

compare elementary C-S ineq:

$$\Big|\sum_k \overline{v}_k w_k\Big|^2 \;\le\; \sum_k |v_k|^2 \;\sum_k |w_k|^2$$

For $a_k > 0$ let $v_k = a_k^{1/2},\; w_k = a_k^{-1/2} x_k$

$$\Big|\sum_k x_k\Big|^2 \;\le\; \sum_k a_k \;\sum_k \overline{x}_k \frac{1}{a_k} x_k$$

Rewrite $\quad \Big(\sum_k \overline{x}_k\Big) \frac{1}{\sum_k a_k} \Big(\sum_k x_k\Big) \;\le\; \sum_k \overline{x}_k \frac{1}{a_k} x_k$

Lieb and Ruskai (1973) proved operator version

$$\Big(\sum_k X_k^*\Big) \frac{1}{\sum_k A_k} \Big(\sum_k X_k\Big) \;\le\; \sum_k X_k^* \frac{1}{A_k} X_k$$

Not suff. for SSA — need Araki rel mod op hidden in L_A and R_B.

Compare proof: $\;\big|\sum_k v_k + t w_k\big|^2 \ge 0 \;\forall\, t\;$ choose t to minimize

Remarks on $q \neq 1 - p$

$p, q > 0,\ p + q < 1 \qquad \mathrm{Tr}\, K^* A^p K B^{1-p}$ concave

Write $\mathrm{Tr}\, K^* A^p K B^q = \mathrm{Tr}\, K^* A^p K (B^s)^{1-p} \qquad 0 < s = \frac{1}{1-p} < 1$

B^s is op monotone and op concave for $s \in (0,1)$

$$\big(\lambda B_1 + (1-\lambda) B_2\big)^s > \lambda B_1^s + (1-\lambda) B_2^s$$

Note: $f(x)$ strictly concave and op concave $\Rightarrow$ strict op ineq

$$\begin{aligned} \mathrm{Tr}\, K^* A^p K B^q &= \mathrm{Tr}\, K^* A^p K \Big[\big(\lambda B_1 + (1-\lambda) B_2\big)^s\Big]^{1-p} \\ &> \mathrm{Tr}\, K^* A^p K \big(\lambda B_1^s + (1-\lambda) B_2^s\big)^{1-p} \\ &\geq \lambda \mathrm{Tr}\, K^* A_1^p K (B_1^s)^{1-p} + (1-\lambda) \mathrm{Tr}\, K^* A_2^p K (B_2^s)^{1-p} \\ &= \lambda \mathrm{Tr}\, K^* A_1^p K B_1^q + (1-\lambda) \mathrm{Tr}\, K^* A_2^p K B_2^q \end{aligned}$$

get equal only for trivial cases, $B_1 = B_2$ or $\lambda = 0, 1$.

Monotonicity under partial traces

Prove MPT: Recall gen Pauli ops,

$$Z|e_n\rangle = e^{2\pi i n/d}|e_n\rangle \qquad X|e_n\rangle = |e_{n+1}\rangle$$

$$\sum_j Z^j A Z^{-j} = d\, A_{\rm diag} \qquad \sum_j X^j A_{\rm diag} X^{-j} = (\operatorname{Tr} A) I$$

$$\tfrac{1}{d}\sum_j \sum_k X^j Z^k\, A\, (X^j Z^k)^* = (\operatorname{Tr} A)\, I$$

$W_n = X_j Z_k$ in some ordering $n = 1, 2, \ldots d^2$, e.g., $n = j + d(k-1)$

$$\tfrac{1}{d_2}\sum_n (I_1 \otimes W_n)\, A_{12}\, (I_1 \otimes W_n)^* = A_1 \otimes I_2$$

Discrete version of Uhlmann's observation that partial trace can be obtained by integrating over $SU(n)$ using Haar measure.

$$J_p(K_2, A_2, B_2) = J_p(I_1 \otimes K_2, \tfrac{1}{d_1} I_1 \otimes A_2, \tfrac{1}{d_1} I_1 \otimes B_2)$$

$$= \tfrac{1}{d_1^2} J_p\Big(K_{12}, \textstyle\sum_n (W_n \otimes I_2) A_{12} (W_n \otimes I_2)^*, \textstyle\sum_n (W_n \otimes I_2) B_{12} (W_n \otimes I_2)^*\Big)$$

$$\leq \tfrac{1}{d_1^2} \sum_n J_p(I_1 \otimes K_2, (W_n \otimes I_2) A_{12} (W_n \otimes I_2)^*, (W_n \otimes I_2) B_{12} (W_n \otimes I_2)^*)$$

$$= \tfrac{1}{d_1^2} \sum_n J_p(I_1 \otimes K_2, A_{12}, B_{12}) = J_p(I_1 \otimes K_2, A_{12}, B_{12})$$

used $J_p(I_1 \otimes K_2, A_{12}, B_{12})$ wrote $K_{12} = I_1 \otimes K_2$

$$= J_p\big(I_1 \otimes K_2, (W_n \otimes I_2) A_{12} (W_n \otimes I_2)^*, (W_n \otimes I_2) B_{12} (W_n \otimes I_2)^*\big)$$

$J_1(I, A_2, B_2) \leq J_1(I, A_{12}, B_{12})$ gives $H(A_2, B_2) \leq H(A_{12}, B_{12})$

Cor: SSA $H(A_{23}, A_2) \leq H(A_{123}, A_{13})$

“no transparent proof of SSA is known”

p. 645 of *Quantum Computation and Quantum Information*

Michael A. Nielsen and Isaac L. Chuang (Cambridge Press, 2000)

based on B. Simon’s version adapted from Uhlmann (1977) of

“elementary” proof of $(A, B) \mapsto \text{Tr}\, K^* A^p K B^{1-p}$ concave

similar argument in Wehrl *Rev. Mod. Phys* (1978). BUT

- MBR, “Lieb’s simple proof of concavity ...” quant-ph/0404126
Int. J. Quant Info. **3**, 579–590 (2005) Schwarz + max mod

- Ando’s argument described in Carlen’s talk

- Petz – uses Δ_{AB} in book; elem version in quant-ph/0408130

- Proof here based on Schwarz ineq. using L_A, R_B really elem.
based on Lesniewski and Ruskai, JMP; and MBR quant-ph/0604206

Equality conditions in $J_p(K, A, X)$ convex

$$\begin{aligned}
\int_0^\infty \operatorname{Tr} K^* A \frac{1}{L_A + tR_B}(AK)\nu(t)dt &\leq \int_0^\infty \sum_j \operatorname{Tr}(A_j K)^* \frac{1}{L_{A_j} + tR_{B_j}}(A_j K)\nu(t) \\
&= \sum_j \int_0^\infty \operatorname{Tr}(A_j K)^* \frac{1}{L_{A_j} + tR_{B_j}}(A_j K)\nu(t)c
\end{aligned}$$

Equal $\Leftrightarrow$ equal for each term in integ , i.e., $M_j = 0 \qquad \forall\ j,\ \forall\ t$

$$(L_{A_j} + tR_{B_j})^{-1}(X_j) = (L_A + tR_B)^{-1}(X) \qquad \forall\ j,\ \forall\ t$$

equality conditions independent of $p \in (0, 2)$

$$X = AK \qquad (I + t\Delta_{A_jB_j}^{-1})^{-1}(K) = (I + t\Delta_{AB}^{-1})^{-1}(K) \qquad \forall\ j,\ \forall\ t$$

$$X = BK \qquad (\Delta_{A_jB_j} + tI)^{-1}(K) = (\Delta_{AB} + tI)^{-1}(K) \qquad \forall\ j,\ \forall\ t$$

Recall $\Delta_{AB} = L_A R_B^{-1} > 0$ prod of commuting pos def ops

$$(\Delta_{A_j B_j} + tI)^{-1}(K) = (\Delta_{AB} + tI)^{-1}(K) \qquad \forall\ j,\ \forall\ t$$

$\Delta_{AB} > 0 \quad \Rightarrow \quad (\Delta_{AB} + tI)^{-1}$ anal cont to $\mathbf{C}\backslash(-\infty, 0]$

can apply Cauchy integral Thm. to get

$\Rightarrow\ G(\Delta_{A_j B_j})(K) = G(\Delta_{AB})(K) \quad \forall\ j \quad G$ anal on $\mathbf{C}\backslash(-\infty, 0]$

allows several useful formulations

$\Rightarrow\ (\Delta_{AB} + tI)$ and $(I + \Delta_{AB}^{-1} t)$ forms equiv.

Equivalent equality conditions

Thm: For fixed K, and $A = \sum_j A_j, B = \sum_j B_j$ TFAE

a) $J_p(K, A, B) = \sum_j J_p(K, A_j, B_j)$ for all $p \in (0, 2)$.

b) $J_p(K, A, B) = \sum_j J_p(K, A_j, B_j)$ for some $p \in (0, 2)$.

c) $(\Delta_{A_j B_j} + tI)^{-1}(K) = (\Delta_{AB} + tI)^{-1}(K) \quad \forall\, j$ and $\forall\, t > 0$.

d) $A_j^{it} K B_j^{-it} = A^{it} K B^{-it} \quad \forall\, j$ and $\forall\, t > 0$.

e) $(\log A - \log A_j)K = K(\log B - \log B_j) \quad \forall\, j$.

In addition when $K = I$, equiv to

f) There are $D_j > 0$ such that $[A_j, D_j] = [B_j, D_j] = 0$, and

$$A_j = AD^{-1}D_j, \quad B_j = BD^{-1}D_j \text{ with } D = \sum_j D_j$$

neccessity of (f) uses sufficient subalgebra – developed by Petz

formulation here from Jenčová and Petz, CMP, **263**, 259–276 (2006).

Equality conditions for SSA

use form $\log A_{123} - \log A_{12} - \log A_{23} + \log A_2 = 0$

Easy to see $A_{123} = A_1 \otimes A_{23}$ or $A_{12} \otimes A_3$ will suffice

If $\mathcal{H}_2 = \mathcal{H}_{2_L} \otimes \mathcal{H}_{2_R}$ then $A_{123} = A_{12_L} \otimes A_{2_R 3}$ will suffice

Thm: Equality holds in SSA if and only if

$$\mathcal{H}_2 = \bigoplus_n \mathcal{H}_n^L \otimes \mathcal{H}_n^R \quad \text{and} \quad A_{123} = \bigoplus_n A_n^L \otimes A_n^R$$

with $A_n^L \in \mathcal{B}(\mathcal{H}_1 \otimes \mathcal{H}_n^L), \quad A_n^R \in \mathcal{B}(\mathcal{H}_n^R \otimes \mathcal{H}_3)$

Cor: Equality in $\operatorname{Tr} A_{23}^p A_2^{1-p} \leq \operatorname{Tr} A_{123}^p A_{12}^{1-p}$ iff same cond

$p \in (0,1) \ \leq \qquad\qquad p \in (1,2) \ \geq$

Carlen Lieb Inequalities

$$\begin{aligned}\widehat{\Upsilon}_{p,1}(K,A) &= \tfrac{1}{(p-1)}\left[\mathrm{Tr}\,K^*A^pK)^{1/p} - \tfrac{1}{p}\mathrm{Tr}\,K^*AK\right] \\ &= \inf\left\{J_p(K,A,X) + \tfrac{1}{p}\mathrm{Tr}\,X : X > 0\right\}\end{aligned}$$

$$\widehat{\Phi}_{p,1}\Big(\sum_k A_k\Big) = \widehat{\Phi}_{p,1}(\mathcal{A}) = \tfrac{1}{(p-1)}\left[\Big(\mathrm{Tr}\sum_k A_k^p\Big)^{1/p} - \tfrac{1}{p}\mathrm{Tr}\sum_k A_k\right]$$

$$\widehat{\Psi}_{p,1}(\mathcal{A}_{12}) = \tfrac{1}{(p-1)}\left[\mathrm{Tr}_1\big(\mathrm{Tr}_2\,\mathcal{A}_{12}^p\big)^{1/p} - \tfrac{1}{p}\mathrm{Tr}_{12}\,\mathcal{A}_{12}\right]$$

All convex for $0 < p \leq 2$. $\widehat{\Phi}(\mathcal{A})$ is block diag case of $\widehat{\Psi}(\mathcal{A}_{12})$

conditional entropy $\widehat{\Psi}_{(1,1)}(\mathcal{A}_{12}) = S(\mathcal{A}_1) - S(\mathcal{A}_{12})$

$$|\mathbb{1}\rangle = (1, 1, \ldots, 1) \qquad |e_1\rangle = (1, 0, \ldots, 0)$$

$$\mathcal{K} = \tfrac{1}{d} I \otimes |\mathbb{1}\rangle\langle e_1| = \begin{pmatrix} I & 0 & \ldots & 0 \\ I & 0 & \ldots & 0 \\ \vdots & \vdots & & \vdots \\ I & 0 & \ldots & 0 \end{pmatrix}$$

$$\mathcal{A} = \sum_k A_k \otimes |e_k\rangle\langle e_k| = \begin{pmatrix} A_1 & 0 & 0 & \ldots & 0 \\ 0 & A_2 & 0 & \ldots & 0 \\ 0 & 0 & A_1 & \ldots & 0 \\ \vdots & & & \ddots & \vdots \end{pmatrix}$$

$$\mathcal{K}^* \mathcal{A}^p \mathcal{K} = \big(\textstyle\sum_k A_k^p\big) \otimes |e_1\rangle\langle e_1| \quad \Rightarrow \quad \widehat{\Phi}_{p,1}(\mathcal{A}) = \widehat{\Upsilon}_{p,1}(K, A)$$

MPT in Carlen-Lieb

monotonicity: $\widehat{\Psi}_{(p,1)}(\mathcal{A}_{23}) \;\le\; \widehat{\Psi}_{(p,1)}(\mathcal{A}_{123})$

conditional entropy: $\widehat{\Psi}_{(1,1)}(\mathcal{A}_{23}) \;=\; S(\mathcal{A}_2) - S(\mathcal{A}_{23})$

$p = 1 \qquad S(\mathcal{A}_2) - S(\mathcal{A}_{23}) \le S(\mathcal{A}_{12}) - S(\mathcal{A}_{123})$ SSA

Carlen-Lieb Minkowski:

$$\begin{aligned}
\mathrm{Tr}_3 \left[\mathrm{Tr}_2(\mathrm{Tr}_1\, \mathcal{A}_{123})^p\right]^{1/p} \;&=\; \Psi_{(p,1)}(\mathcal{A}_{32}) \\
&\le\; \Psi_{(p,1)}(\mathcal{A}_{132}) \;=\; \mathrm{Tr}_3\, \mathrm{Tr}_1 \big(\mathrm{Tr}_2 \mathcal{A}_{123}^p\big)^{1/p}
\end{aligned}$$

for $1 < p \le 2$ and reverse ineq $\ge$ for $0 < p < 1$.

$p = 1$ equality conditions same as for SSA

Equality conditions in Carlen-Lieb

Extend equal conds to all $p \in (0,2)$ by Cor to SSA equal conds

Rough idea: Equal in SSA $\Rightarrow$ equal in MPT proof

$\Rightarrow$ equal in convexity for $J_p(I, A_{12}, I_1 \otimes A_2)$

can actually use SSA equal conds to improve this

equal in $\widehat{\Phi}(\mathcal{A})$ convex $\Leftrightarrow$ equal in $J_p(I, \mathcal{A}, \mathrm{Tr}_2(\mathcal{A}) \otimes I_2)$

equal in $\widehat{\Psi}(\mathcal{A}_{12})$ convex $\Leftrightarrow$ equal in $J_p(I, \mathcal{A}_{123}, \mathcal{A}_1 \otimes I_{23})$

$$\mathcal{A}_{123} = \sum_n (I \otimes W_n)\mathcal{A}_{12}(I \otimes W_n)^* \otimes |e_n\rangle\langle e_n|$$

equal in "new" SSA from $\widehat{\Psi}_{(p,1)}(\mathcal{A}_{23}) \leq \widehat{\Psi}_{(p,1)}(\mathcal{A}_{123})$ indep of p

Generalizations of SSA

Uniform treatment led to two distinct generalizations of SSA

pseudo p-metric based on MPT of $J_p(1, A_{123}, A_{23})$

$$\operatorname{Tr} A_{23}^p A_2^{1-p} \begin{array}{c} \leq \\ \geq \end{array} \operatorname{Tr} A_{123}^p A_{12}^{1-p} \quad \begin{array}{l} p \in (0,1) \\ p \in (1,2) \end{array}$$

pseudo p-norm based on MPT of $\widehat{\Psi}(\mathcal{A}_{123})$

$$\operatorname{Tr}_2(\operatorname{Tr}_3 \mathcal{A}_{23}^p)^{1/p} \begin{array}{c} \geq \\ \leq \end{array} \operatorname{Tr}_{12}(\operatorname{Tr}_3 \mathcal{A}_{123}^p)^{1/p} \quad \begin{array}{l} p \in (0,1) \\ p \in (1,2) \end{array}$$

Compare Renyi $\frac{1}{1-p} \log \operatorname{Tr} A^p$ and Tsallis $\frac{1}{p-1}(1 - \operatorname{Tr} A^p)$ entropy